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SÉRIE:
RECHERCHES SUR LES DÉFORMATIONS

Volume LXV, no. 1

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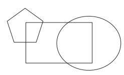

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References

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Recherches sur les déformations no. 1

pp. 9–20

In memory of
Professor Zygmunt Charzyński (1914–2001)

Grzegorz Oleksik

ON THE NONDEGENERATE JUMPS OF THE ŁOJASIEWICZ EXPONENT

Summary

The aim of this paper is to give formulas for jumps upwards and downwards of the Łojasiewicz exponent in nondegenerate deformations of a curve singularity f in terms of its Newton diagram.

Keywords and phrases: isolated singularity, deformation, jump, nondegeneracy in the Kouchnirenko sense, Łojasiewicz exponent, Milnor number

1. Introduction

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be an isolated singularity. We define the number

$$\mathcal{L}_0(f) := \inf\{\alpha \in \mathbb{R}_+ : \exists_{C>0}\exists_{r>0}\forall_{\|z\|<r}\|\nabla f(z)\| \geq C\|z\|^\alpha\}$$

and call it the *Łojasiewicz exponent* of f. In [Tes77], B. Teissier calculated $\mathcal{L}_0(f)$ in terms of polar invariants of the singularity f and proved that the Łojasiewicz exponent is lower semicontinuous in any μ-constant deformation of the singularity f. A. Płoski generalized his result and proved that the Łojasiewicz exponent is lower semicontinuous in any multiplicity-constant deformation of a finite holomorphic map germ (see [P11]). Teissier also showed that without the μ-constancy assumption the Łojasiewicz exponent is neither uppern or lower semicontinuous (see [Tes78]). The "jump phenomena" of the Łojasiewicz exponent were rediscovered by some authors (see [MN05]). By the *jump downards* of $\mathcal{L}_0(f)$ we mean the minimum non-zero positive difference between the Łojasiewicz exponent of f and one of its deformations (f_s). We define in analogous way the *jump upwards* (see Section 3). We give formulas for jumps upwards and downwards of the Łojasiewicz exponent in nondegenerate deformations of a curve singularity f in terms of its Newton diagram and the Farey's fractions (see Theorems 4.1, 4.5). We also give some examples showing that a nondegenerate jump of the Łojasiewicz exponent of f may be different from a jump of the Łojasiewicz exponent of f or be the same in some cases (see Examples 4.4, 4.3).

2. Preliminaries

2.1. Deformations of singularities

Let $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$ be a non-zero holomorphic function in an open neighbourhood of $0 \in \mathbb{C}^n$. We say that f is *an isolated singularity* if f has an isolated critical point at the origin i.e. $f(0) = 0$, $\nabla f(0) = 0$ and $\nabla f(z) \neq 0$ for $z \neq 0$ near $0 \in \mathbb{C}^n$, where $\nabla f = (f'_{z_1}, \dots, f'_{z_n})$.

We say that a holomorphic function $F = F(z, s) : (\mathbb{C}^n \times \mathbb{C}, 0) \longrightarrow (\mathbb{C}, 0)$ is a *deformation* of f if

1. $F(z, 0) = f(z)$
2. $F(\cdot, s)$ is an isolated singularity for every s.

We shall write $F(z, s) = f_s(z)$.

2.2. The Newton diagram and nondegenerate singularities

We denote

$$\mathbb{N} = \{0, 1, 2, \dots\}, \quad \mathbb{N}_+ = \mathbb{N} \setminus \{0\} \quad \text{and} \quad \mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}.$$

Let $\sum_{\nu \in \mathbb{N}^n} a_\nu z^\nu$ be the Taylor expansion of f at 0. We define the set $\operatorname{supp} f = \{\nu \in \mathbb{N}^n : a_\nu \neq 0\}$ and call it *the support of* f.

We define

$$\Gamma_+(f) = \operatorname{conv}\{\nu + \mathbb{R}^n_+ : \nu \in \operatorname{supp} f\} \subset \mathbb{R}^n$$

and call it *the Newton diagram of* f. Let $u \in \mathbb{R}^n_+ \setminus \{0\}$. Put

$$l(u, \Gamma_+(f)) = \inf\{\langle u, v\rangle : v \in \Gamma_+(f)\},$$
$$\Delta(u, \Gamma_+(f)) = \{v \in \Gamma_+(f) : \langle u, v\rangle = l(u, \Gamma_+(f))\}.$$

We say that $S \subset \mathbb{R}^n$ is *a face of* $\Gamma_+(f)$ if $S = \Delta(u, \Gamma_+(f))$ for some $u \in \mathbb{R}^n_+ \setminus \{0\}$. The vector u is called *a primitive vector of* S. It is easy to see that S is a closed and convex set and $S \subset \operatorname{Fr}(\Gamma_+(f))$, where $\operatorname{Fr}(A)$ denotes the boundary of A. One may check that a face $S \subset \Gamma_+(f)$ is compact if and only if there exists a primitive vector of S which has all coordinates positive. We call the family of all compact faces of $\Gamma_+(f)$ *the Newton boundary of* f and denote it by $\Gamma(f)$. We denote by $\Gamma^k(f)$ the set of all compact k-dimensional faces of $\Gamma(f)$, $k = 0, \ldots, n-1$. For every compact face $S \in \Gamma(f)$ we define the polynomial

$$f_S = \sum_{\nu \in S} a_\nu z^\nu.$$

We say that f is *nondegenerate on the face* $S \in \Gamma(f)$ if the system of equations

$$\frac{\partial f_S}{\partial z_1} = \ldots = \frac{\partial f_S}{\partial z_n} = 0$$

has no solution in $(\mathbb{C}^*)^n$, where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. We say that f is *nondegenerate in the sense of Kouchnirenko* (shortly *nondegenerate*) if it is nondegenerate on each face of $\Gamma(f)$ (see [Ko76]).

We say that f is *nearly convenient* if the distance of $\Gamma_+(f)$ to every coordinate axis does not exceed 1.

We say that $S \in \Gamma^{n-1}(f) \subset \mathbb{R}^n$ is *an exceptional face with respect to the axis* OX_i if one of its vertices is at distance 1 to the axis OX_i and another vertices constitute $(n-2)$-dimensional face which lies in one of the coordinate hyperplane including the axis OX_i.

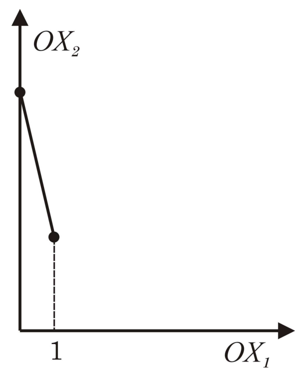

Fig. 1: A 1-dimensional exceptional face with respect to the axis OX_2.

We say that $S \in \Gamma^{n-1}(f)$ is *an exceptional face of* f if there exists $i \in \{1, \ldots, n\}$ such that S is an exceptional face with respect to the axis OX_i. Denote by E_f the set of exceptional faces of f.

Denote by $x_i(P)$, $i = 1, \ldots, n$ the i-coordinate of the point $P \in \mathbb{R}^n$. For every face $S \in \Gamma^{n-1}(f)$ we shall denote by $x_1(S), \ldots, x_n(S)$ the coordinates of the intersection of the hyperplane determined by S with the coordinate axes. We define $m(S) := \max\{x_1(S), \ldots, x_n(S)\}$. It is easy to see that

$$x_i(S) = l(u, \Gamma_+(f))/u_i,\ i = 1, \ldots, n,$$

where u is a primitive vector of S. If f is a nondegenerate isolated curve singularity we may easily read off the Łojasiewicz exponent from the $\Gamma_+(f)$.

Theorem 2.1. [Len96] *Let $f : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ be an isolated singularity nondegenerate in Kouchnirenko's sense and $\Gamma(f) \setminus E_f \neq \emptyset$. Then*

$$\text{(1)} \qquad \pounds_0(f) = \max_{S \in \Gamma(f) \setminus E_f} m(S) - 1.$$

2.3. Farey's fraction

The sequence of Farey's fractions of order N is the increasing sequence F_N of all fractions p/q, $1 \leq q \leq N$, $0 \leq p \leq q$, where p, q are relatively prime integers.

Example 2.2. $F_4 = \left(\frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}\right)$.

To determine the sequence of Farey's fractions we may use the following classical result from number theory.

Theorem 2.3. [Cauchy-Farey] *Let F_N be the sequence of Farey's fractions of order N. If*

$$F_N(n) = \frac{a}{b} \quad \textit{and} \quad F_N(n+1) = \frac{c}{d} \quad \textit{then} \quad bc - ad = 1.$$

3. Auxiliary results

Denote by $\mathcal{O}_n$ the local ring of germs of holomorphic functions in n-variables at $0 \in \mathbb{C}^n$. Let $f, g \in \mathcal{O}_n$.

We say that *f and g are topologically right-left equivalent* ($f \overset{topRL}{\sim} g$) if there exist homeomorphism germs $\phi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ and $\psi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $f = \phi \circ g \circ \psi$.

Let us recall that *the Milnor number of an isolated singularity* $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$ is defined as

$$\mu_0(f) = \dim_{\mathbb{C}} \mathcal{O}^n/(f'_{z_1}, \ldots, f'_{z_n}).$$

From now on, we shall restrict our discussion to the case of a curve singularity.

Proposition 3.1. *Let $f : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ be an isolated singularity and let (f_s) be a deformation of f. There exists some neighbourhood U of $0 \in \mathbb{C}$, such that*

$$\pounds_0(f_s) = \text{const}$$

for every $s \in U \setminus \{0\}$.

Proof. Let (f_s) be a deformation of f. By the upper semicontinuity of the Milnor number in the Zariski topology (see [GLS, Thm. 2.6]) we have $\mu_0(f) \geq \mu_0(f_s)$ and $\mu_0(f_s) = \mathit{const}$ for $s \in U \setminus \{0\}$, where U is a neighbourhood of $0 \in \mathbb{C}$. Let $s_0 \in U \setminus \{0\}$ and $U_{s_0} \subset U \setminus \{0\}$ be some neighbourhood of s_0. Then $\mu_0(f_s) = \mu_0(f_{s_0})$ for $s \in U_{s_0}$. Hence by Le-Ramanujan Theorem $f_{s_0} \overset{\text{top}RL}{\sim} f_s$. Since the Łojasiewicz exponent is a topological invariant for $n = 2$ (see [Tes77]) we get $\L_0(f_{s_0}) = \L_0(f_s)$. Therefore $\L_0(f_s)$ is locally constant and since $U \setminus \{0\}$ is connected we have $\L_0(f_s) = \text{const}$ for every $s \in U \setminus \{0\}$. □

As a consequence of the above proposition we may correctly define the *Łojasiewicz exponent of a deformation* (f_s) of f as $\L_0((f_s)) = \L_0(f_{s_0})$, $s_0 \in U \setminus \{0\}$, where U is a suitable neighbourhood of $0 \in \mathbb{C}$.

Let $L(f)$ be the set of Łojasiewicz exponents of all deformations of f. The next proposition gives upper bound of the cardinality of $L(f)$ and shows that $L(f)$ is a finite set.

Proposition 3.2.

$$\#L(f) \leq 1 + \sum_{k=2}^{\mu_0(f)} \phi(k),$$

where ϕ is the Euler function.

Proof. Let (f_s) be a deformation of f. By Płoski result [P85, Prop. 1.2] we have

$$\L(f_s) = \frac{p}{q},$$

where p, q are relatively prime integers such that $1 \leq q \leq p \leq \mu_0(f_s)$. On the other hand by the upper semicontinuity of the Milnor number we have $\mu_0(f_s) \leq \mu_0(f)$. Summing up

$$\#L(f) \leq 1 + \sum_{k=2}^{\mu_0(f)} \phi(k).$$

It finishes the proof. □

Some examples suggest a better estimate than in Proposition 3.2, so we put the following conjecture.

Conjecture 3.3. $\#L(f) \leq \mu_0(f)$.

Denote by $\mathcal{D}(f)$ the family of all deformations of an isolated singularity f. Consider the following subsets of $\mathcal{D}(f)$

$$\mathcal{D}^+(f) = \{(f_s) \in \mathcal{D}(f) : \L_0((f_s)) > \L_0(f)\}$$
$$\mathcal{D}^-(f) = \{(f_s) \in \mathcal{D}(f) : \L_0((f_s)) < \L_0(f)\}.$$

By Proposition 3.2 the set $L(f)$ is finite. Hence we can correctly define the following notions. Let $\mathcal{D}^+(f) \neq \emptyset$. Define the number

$$\delta^+(f) = \min\{\L_0((f_s)) - \L_0(f) : (f_s) \in \mathcal{D}^+(f)\}$$

and call it *the jump upwards of the Łojasiewicz exponent of f*. If $\mathcal{D}^+(f) = \emptyset$ we put $\delta^+(f) = 0$.

Let $\mathcal{D}^-(f) \neq \emptyset$. Define analogously to $\delta^+(f)$ the number

$$\delta^-(f) = \min\{\pounds_0(f) - \pounds_0((f_s)) : (f_s) \in \mathcal{D}^-(f)\}$$

and call it *the jump downwards of the Łojasiewicz exponent of* f. If $\mathcal{D}^-(f) = \emptyset$ we put $\delta^-(f) = 0$. Denote by $\mathcal{D}_{nd}(f)$ the family of all nondegenerate deformations of an isolated singularity f. We define analogously the sets $\mathcal{D}^+_{nd}(f)$, $\mathcal{D}^-_{nd}(f)$ and *the nondegenerate jump upwards* $\delta^+_{nd}(f)$ *and jump downwards* $\delta^-_{nd}(f)$ *of the Łojasiewicz exponent of* f.

4. Main result

Let $f : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ be an isolated singularity. It is easy to check that f is nearly convenient. Then for every $i \in \{1, 2\}$ there exists a vertex on the axis OX_i or at distance 1 from the axis OX_i.

Let $K \subset \{1, 2\}$ be the set of the indices i satisfying two following conditions:

a) There exists the vertex $A_i \in \Gamma^0(f)$ at distance 1 from the axis OX_i such that $\pounds_0(f) < 2x_i(A_i) - 1$;

b) If $OX_i \cap \Gamma^0(f) \neq \emptyset$ then $\pounds_0(f) < x_i(OX_i \cap \Gamma^0(f)) - 1$.

If $K \neq \emptyset$, we put $M = \max_{i\in K} x_i(A_i)$. Let r be the fraction before

$$\frac{1}{\pounds_0(f) + 1 - M}$$

in the sequence F_M. Now, we give the formula for nondegenerate jump upwards of the Łojasiewicz exponent of f.

Theorem 4.1. *Let* $f : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ *be an isolated and nondegenerate singularity. Then*

a) *If* $K = \emptyset$ *then* $\delta^+_{nd}(f) = 0$.

b) *If* $K \neq \emptyset$ *then*

$$\delta^+_{nd}(f) = M + \frac{1}{r} - \pounds_0(f) - 1.$$

Proof. a) Let (f_s) be a deformation of f. By Theorem 2.1 we see that to change $\pounds_0(f)$ it is necessary to deform f by adding monomials related to the points below $\Gamma_+(f)$. We have $\Gamma_+(f_s) = \text{const}$ for sufficiently small $s \neq 0$. Since $K = \emptyset$, by Theorem 2.1 using elementary geometry methods we easily get

$$\pounds_0((f_s)) \leq \pounds_0(f).$$

Therefore $\delta^+_{nd}(f) = 0$ in this case.

b) It is easy to check that

$$\Gamma^1(f) \setminus E_f = \emptyset \quad \text{if and only if} \quad \Gamma^0(f) = \{(1, 1)\}.$$

Since $K \neq \emptyset$ and $\pounds_0(f) \geq 1$, then $\Gamma^0(f) \neq \{(1,1)\}$. Hence $\Gamma^1(f) \setminus E_f \neq \emptyset$.

Let $i \in K$. Then there exists a vertex $A_i \in \Gamma^0(f)$ at distance 1 from the axis OX_i, such that

$$\pounds_0(f) < 2x_i(A_i) - 1.$$

Let $Q \in \mathbb{R}^2$ be such that $Q \neq A_i$ and $\overline{A_iQ} \not\parallel OX_i$. Denote by $x_i(\overline{A_iQ})$ the intersection coordinate of the line determined by the segment $\overline{A_iQ}$ and the axis OX_i.

Let D_i be the orthogonal projection of A_i to OX_{3-i} and l be the line passing through A_i and the point on axis OX_i with i-coordinate equal to $\pounds_0(f) + 1$. Put $C_i = l \cap OX_{3-i}$ and define the set $\triangle_i = \triangle A_iC_iD_i$.

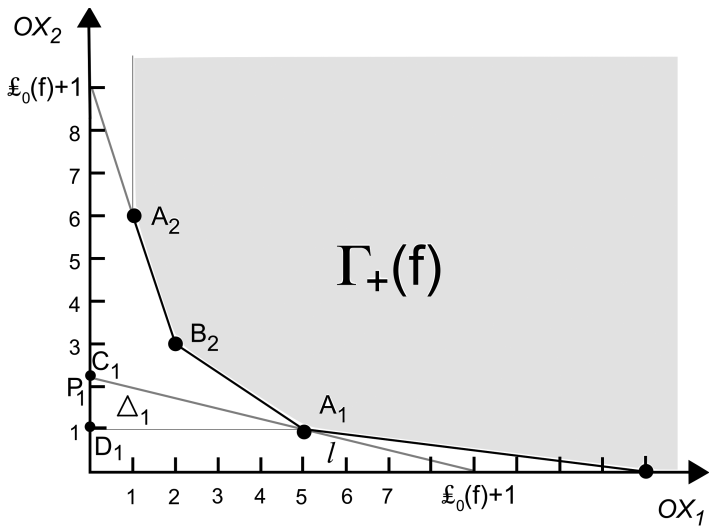

Fig. 2: $f(x,y) = x^5y + x^2y^3 + xy^6 + x^{13}$, $K = \{1,2\}$, $M = 6$, $\pounds_0(f) = 8$.

Put

$$S_i = \{Q \in \triangle_i \cap \mathbb{N}^2 : x_i(\overline{A_iQ}) > \pounds_0(f) + 1\}.$$

Let $e_1 = (1,0), e_2 = (0,1)$. Define the point $P_i = D_i + e_{3-i}$. We have

$$x_i(\overline{A_iP_i}) = 2x_i(A_i) > \pounds_0(f) + 1. \quad (2)$$

Hence $P_i \in S_i$ and $S_i \neq \emptyset$. It is easy to check that

$$S_i = (\triangle_i \cap \mathbb{N}^2) \setminus (\overline{A_iC_i} \cap \overline{A_iD_i}).$$

Put $u_i = \min_{Q \in S_i} x_i(\overline{A_iQ})$.

For every $Q = (q_1, q_2)$ in S_i we define the deformation $(f_s^Q) \in \mathcal{D}_{nd}$,

$$f_s^Q(x,y) = f(x,y) + sx^{q_1}y^{q_2} + sx^{\lceil x_1(C_i) \rceil}y^{\lceil x_2(C_i) \rceil},$$

where $\lceil a \rceil$ is the smallest integer not less than a. We calculate

$$£_0((f_s^Q)) = x_i(\overline{A_iQ}) - 1 > £_0(f).$$

Hence $(f_s^Q) \in \mathcal{D}_{nd}^+$. Conversely by definition of the sets $S_i, i \in K$ and Theorem 2.1 for every deformation $(f_s) \in \mathcal{D}_{nd}^+$ there exists $i \in K$ and $Q \in S_i$ such that

$$£_0((f_s)) = x_i(\overline{A_iQ}) - 1.$$

Summing up

$$\min\{£_0((f_s)) : (f_s) \in \mathcal{D}_{nd}^+(f)\} = \min_{i \in K} u_i - 1. \tag{3}$$

Using elementary geometry methods we calculate

$$u_i = \min\left\{x_i(A_i) + \frac{n}{m} : x_i(A_i) + \frac{n}{m} > £_0(f) + 1, m, n \in \mathbb{N}_+, n \leq x_i(A_i)\right\}, i \in K.$$

Put

$$u = \min\left\{M + \frac{n}{m} : M + \frac{n}{m} > £_0(f) + 1, m, n \in \mathbb{N}_+, n \leq M\right\}.$$

By Theorem 2.1 we easily get $£_0(f) \geq M$ and we calculate

$$u = \min_{i \in K} u_i. \tag{4}$$

Define

$$r = \max\left\{\frac{m}{n} : \frac{m}{n} < \frac{1}{£_0(f) + 1 - M}, m, n \in \mathbb{N}_+, n \leq M\right\}.$$

Then

$$u = M + \frac{1}{r}. \tag{5}$$

We show that $1/(£_0(f) + 1 - M)$ is the Farey's fraction of order M and then by definition r is the one before. Since $£_0(f) \geq M$, we get

$$1/(£_0(f) + 1 - M) \leq 1.$$

If $£_0(f)$ is an integer, then by definition of K and M we get

$$£_0(f) + 1 - M < M \quad \text{and} \quad 1/(£_0(f) + 1 - M) \in F_M.$$

If $£_0(f)$ is not an integer, then by Theorem 2.1 there exists $j \in \{1, 2\}$ such that

$$£_0(f) = x_j(\overline{A_jB_j}) - 1, \quad \text{where} \quad \overline{A_jB_j} \in \Gamma^1(f)$$

is the unexceptional segment with vertex A_j. By definition of K we get that $j \in K$. Without loss of generality we may suppose that $£_0(f) = x_2(\overline{A_2B_2}) - 1$ (as in Figure 2). We calculate

$$x_2(\overline{A_2B_2}) = x_2(A_2) + \frac{x_2(A_2) - x_2(B_2)}{x_1(B_2) - x_1(A_2)} = x_2(A_2) + \frac{n}{m},$$

where $n = x_2(A_2) - x_2(B_2)$ and $m = x_1(B_2) - x_1(A_2)$. Hence

$$\frac{1}{£_0(f) + 1 - M} = \frac{1}{x_2(A_2) + \frac{n}{m} - M} = \frac{m}{n - m(M - x_2(A_2))}.$$

Therefore the latter denominator is less than $n = x_2(A_2) - x_2(B_2) \le M$. Summing up $1/(\pounds_0(f)+1-M)$ is the Farey's fraction of order M. By (3),(4) and (5) we get

$$\delta^+_{nd}(f) = u - 1 - \pounds_0(f) = M + \frac{1}{r} - \pounds_0(f) - 1.$$

$\square$

Example 4.2. Let

$$f(x,y) = x^5y + x^2y^3 + xy^6 + x^{13}$$

(see Figure 2). By Theorem 2.1 we calculate $\pounds_0(f) = 8$. We check that $K = \{1, 2\}$ and $M = 6$. Now, we calculate

$$1/(\pounds_0(f) + 1 - M) = 1/3$$

and by Cauchy-Farey Theorem we get $r = 1/4$. Hence by Theorem 4.1(b) we have

$$\delta^+_{nd}(f) = M + \frac{1}{r} - \pounds_0(f) - 1 = 1.$$

Example 4.3. Let $f(x,y) = y^4 + yx^5$. In this case we get that $K = \{1\}$. Using Theorem 2.1 and Theorem 4.1b) we calculate

$$\pounds_0(f) = 5\frac{2}{3},\ \delta^+_{nd}(f) = \frac{1}{3}.$$

By the result of Płoski-Barosso ([P88], [GaP]) no number in the interval $(5\frac{2}{3}, 6)$ can be the Łojasiewicz exponent of an isolated curve singularity. Therefore in this case we get

$$\delta^+(f) = \delta^+_{nd}(f) = \frac{1}{3}.$$

Now, we give an example of singularity such that the nondegenerate jump upwards of its Łojasiewicz exponent is indeed smaller then the jump upwards of its Łojasiewicz exponent.

Example 4.4. Let

$$f(x,y) = x^3 + y^3.$$

We easily calculate that $\pounds_0(f) = 2$. We check that $K = \emptyset$ and then by Theorem 4.1 (a) we have $\delta^+_{nd}(f) = 0$. On the other hand if we take degenerate deformation

$$f_s(x,y) = f(x,y) + s(x^2 + 2xy + y^2),$$

we get $\pounds_0(f_s) = 3$ and one may check that $\delta^+(f) = 1$. Hence in this case $\delta^+_{nd}(f) < \delta^+(f)$.

Now, we consider the case of the nondegenerate jump downstairs of the Łojasiewicz exponent. First notice that if

$$\Gamma^1(f) \setminus E_f = \emptyset \quad \text{then} \quad \pounds_0(f) = 1 \quad \text{and} \quad \delta^-_{nd}(f) = \delta^-(f) = 0.$$

Suppose that

$$\Gamma^1(f) \setminus E_f \neq \emptyset.$$

By Theorem 2.1 there exists $i \in \{1,2\}$ and the vertices $Q_i, R_i \in \Gamma^0(f)$ such that $\overline{Q_iR_i} \in \Gamma^1(f) \setminus E_f$ and

$$£_0(f) = x_i(\overline{Q_iR_i}) - 1. \tag{6}$$

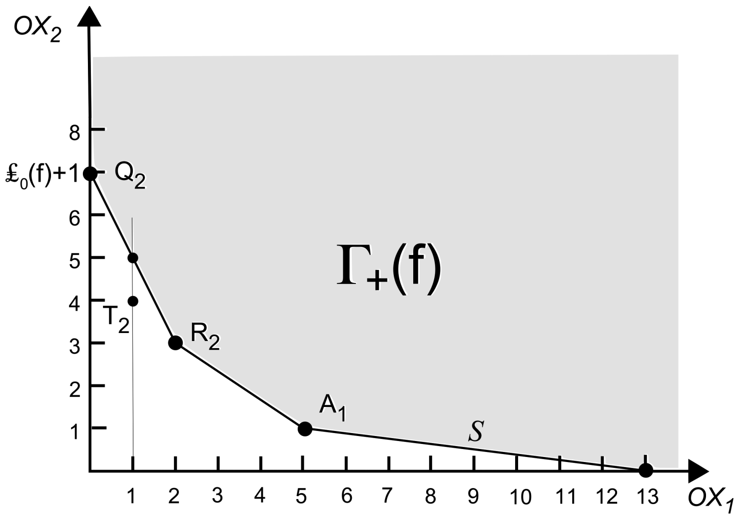

Fig. 3: $f(x,y) = y^7 + x^2y^3 + x^5y + x^{13}$, $I = \{2\}$, $M_1 = 5$, $£_0(f) = 6$.

Let $I \subset \{1,2\}$ be the set of indexes i satisfying (6). Let

$$T_i \in \{x_{3-i} = 1\} \setminus \Gamma_+(f)$$

be the nearest point on the lattice next to

$$\overline{Q_iR_i} \cap \{x_{3-i} = 1\},\ i \in I \quad \text{and put} \quad M = \max_{i \in I} x_i(T_i).$$

In the sequence F_M let r be the next fraction after

$$\frac{1}{£_0(f) + 1 - M}.$$

If $i \notin I$ and

$$\{A_i\} = \{x_{3-i} = 1\} \cap \Gamma_0(f) \neq \emptyset,$$

we put $M_i = \max\{x_i(A_i), x_{3-i}(T_{3-i})\}$ Let r_i be the next fraction after $\frac{1}{£_0(f)+1-M_i}$ in the sequence F_{M_i}.

Using similar methods as in the proof of Theorem 4.1 one may prove the following

Theorem 4.5. *Let $f : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ be an isolated and nondegenerate singularity and $\Gamma^1(f) \setminus E_f \neq \emptyset$. Then*

a) *If* $I = \{1,2\}$ *then* $\delta^-_{nd}(f) = \pounds_0(f) + 1 - M - \frac{1}{r}$,

b) *If* $I = \{i\}$ *for some* $i \in \{1,2\}$ *then we consider two following cases*

b1) *There are no any exceptional segments with respect to the axis* OX_i.
If $\{x_i = 1\} \cap \Gamma_0(f) = \emptyset$, *then* $\delta^-_{nd}(f) = \pounds_0(f) + 1 - M - \frac{1}{r}$
If $\{x_i = 1\} \cap \Gamma_0(f) \neq \emptyset$, *then* $\delta^-_{nd}(f) = \pounds_0(f) + 1 - M_i - \frac{1}{r_i}$,

b2) *There is an exceptional segment* S *with respect to the axis* OX_{3-i}.
If $x_{3-i}(S \cap OX_{3-i}) \geq \pounds_0(f) + 1$, *then* $\delta^-_{nd}(f) = \pounds_0(f) + 1 - M_i - \frac{1}{r_i}$
If $x_{3-i}(S \cap OX_{3-i}) < \pounds_0(f) + 1$, *then* $\delta^-_{nd}(f) = \pounds_0(f) + 1 - M - \frac{1}{r}$.

Example 4.6. Let
$$f(x,y) = y^7 + x^2y^3 + x^5y + x^{13}.$$
Applying Theorem 2.1 we calculate $\pounds_0(f) = 6$. We check that
$$I = \{2\} \quad \text{and} \quad M_1 = \max\{x_1(A_1), x_2(T_2)\} = 5$$
(see Figure 3). Now, we calculate
$$1/(\pounds_0(f) + 1 - M_1) = 1/2$$
and by Cauchy-Farey Theorem we get that $r_1 = 3/5$. We also check that
$$x_1(S \cap OX_1) = 13 \geq \pounds_0(f) + 1.$$
Hence by Theorem 4.5(b2) we have
$$\delta^-_{nd}(f) = \pounds_0(f) + 1 - M_1 - \frac{1}{r_1} = \frac{1}{3}.$$

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References

[GaP] E. García Barosso and A. Płoski A.: *On the Łojasiewicz numbers,* C. R. Acad. Sci. Paris Sér. I **336** (2003), 585–588.

[GLS] G.-M. Greuel, C. Lossen, and E. Shustin, *Introduction to singularities and deformations,* Springer Monographs in Mathematics, Springer, Berlin 2007.

[Ko76] A. G. Kouchnirenko, *Polyèdres de Newton et nombres de Milnor,* Invent. Math. **32** (1976), 1–31.

[Len96] A. Lenarcik, *On the Łojasiewicz exponent of the gradient holomorphic function,* Banach Center Publications **44** (1998), 149–166.

[MN05] J. Mc Neal and A. Némethi, *The order of contact of a holomorphic ideal in* $\mathbb{C}^2$, Math. Z. **250** (2005), 873–883.

[P85] A. Płoski, *Sur l'exposant d'une application analytique II,* Bull. Pol. Acad. Sci. Math. **33** (1985), 123–127.

[P88] A. Płoski, *Multiplicity and the Łojasiewicz exponent*, in: *Singularities*, Banach Center Publ. **20** (1988), 353–364.

[P11] A. Płoski, *Semicontinuity of the Łojasiewicz exponent*, Univ. Iagel. Acta Math. **48** (2011) 103–110.

[Tes77] B. Teissier, *Variétés polaires I – Invariant polaires de singularités d'hypersurfaces*, Invent. Math. **40** (1977), 267–292.

[Tes78] B. Teissier, *Jacobian Newton polyhedra and equisingularity*, Preceedings R.I.M.S. Conference on singularities, Kyoto, April 1978, (Publ. R.I.M.S. 1978).

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O NIEZDEGENEROWANYCH SKOKACH WYKŁADNIKA ŁOJASIEWICZA

S t r e s z c z e n i e
W pracy podajemy wzory na skok w górę i w dół wykładnika Łojasiewicza w klasie niezdegenerowanych deformacji osobliwości krzywej w terminach jej diagramu Newtona.

Słowa kluczowe: osobliwość izolowana, deformacje holomorficzne, skok, niedegeneracja w sensie Kusznirenki, wykładnik Łojasiewicza, liczba Milnora

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Jerzy Grzybowski and Ryszard Urbański

MODELING CRYSTAL GROWTH: POLYHEDRA WITH FACES PARALLEL TO PLANES FROM A FIXED FINITE SET

Summary

In this paper we present growing crystal as a point in finite dimentional space moving inside a cone $\mathcal{S}_G \subset \mathbb{R}^m$ corresponding to the family $\mathcal{B}_G$ of polyhedra with faces parallel to planes from a fixed finite set G. For this purpose we modify Minkowski addition of convex sets and explain in depth a representation of a growth of certain crystal of NaCl. We also provide a number of illustrations of a respective cone $\mathcal{S}'_G$ of symmetric polyhedra and the trajectory of our growing crystal which is a broken line.

Keywords and phrases: crystal growth, abstract cone of convex polyhedra, modified Minkowski addition, trajectory of growing crystal

1. Introduction

Natural geometrical representation of monocrystal is a nonempty bounded closed convex subset of three-dimensional space $\mathbb{R}^3$. The family of non-empty bounded closed convex subsets in Hausdorff topological vector spaces X was given a lot of attention. In particular this family with Minkowski addition

$$A\dot{+}B = \overline{A+B} = \overline{\{a+b \,|\, a \in A, B \in B\}}$$

and multiplication by nonnegative number $\lambda A = \{\lambda a \,|\, a \in A\}$ is an abstract convex cone with ordering by inclusion and with order law of cancellation ($A\dot{+}B \subset B\dot{+}C$ implies $A \subset C$).

Minkowski addition and subtraction

$$A\dot{-}B = \{x \in X \,|\, x + B \subset A\}$$

allow to model crystal growth with the help of one variable multifunction with convex values. As we proved in [1] the formula

$$A(u) = \frac{u-s}{t-s}A(t) \dot{-} \frac{u-t}{t-s}A(s),$$

$s < t < u$ represents the growth of crystal where each face grows with constant velocity.

Minkowski duality enables representation of convex bodies with the help of support functions [4], i.e. for compact convex subset A of $\mathbb{R}^n$ the function $h : \mathbb{R}^n \longrightarrow \mathbb{R}$ defined by $h(\cdot) = \max_{x \in A}\langle \cdot, x\rangle$ is sublinear, it is called *support function* of the set A and denoted by h_A. On the other hand, the set A is equal to $\{x | \langle \cdot, x\rangle \leq h\}$, it is called a *subdifferential* of the sublinear function h at 0 and denoted by $\partial h|_0$.

In reality a crystal of a specific chemical compound has finite number of faces and planes of these fases are determined by its crystalographic structure. Therefore, it is possible to restrict ourselves to the study of a subfamily of convex polyhedra with normal vectors belonging to some fixed finite subset of Euclidean unit sphere.

In Section 2 we modify Minkowski addition and duality for convex polyhedra with normal vectors belonging to some fixed finite set G. In Section 3 we restrict ourselves to symmetric polyhedra and give an example of a family of symmetric polyhedra with faces parallel to 26 faces of truncated cuboctahedron (Example 3.1). In Example 3.2 we show a trajectory of one possible growing crystal of NaCl.

In Section 4 we present Minkowski difference as a projection of nonnegative octant on the cone $\mathcal{B}_G$ of convex polyhedra with normal vectors belonging to G. Minkowski difference and the projection preserves our formulas of crystal growth (see [1]). In Example 4.1 we show that our growing crystal of NaCl is a projection of a uniform motion along a straight line on the cone $\mathcal{B}'_G$ of symmetric convex polyhedra with normal vectors belonging to G.

2. Family of convex polyhedra with faces parallel to planes from fixed finite set

Let $G = \{z_1, ..., z_m\}$ be such a finite subset of the Euclidean unit sphere S^2 that $0 \in \operatorname{int}\operatorname{conv}G$.

Let $\mathcal{B}(\mathbb{R}^3)$ be a family of all nonempty bounded closed convex subsets of $\mathbb{R}^3$. The family $\mathcal{B}(\mathbb{R}^3)$ with Minkowski addition and multiplication by nonnegative numbers is an abstract convex cone satisfying the order law of cancellation. The cone $\mathcal{B}(\mathbb{R}^3)$ can be embeded into a vector space $\mathrm{MRH}(\mathbb{R}^3)$ of virtual bodies, which is a quotient space $\mathcal{B}(\mathbb{R}^3)^2/_\sim$, where $(A, B) \sim (C, D)$ if and only if $A + D = B + C$, and $[A, B]$ is a quotient class of (A, B).

For $A \in \mathcal{B}(\mathbb{R}^3)$ we define a *support function* h_A on $\mathbb{R}^3$ by $h_A(x) = \max_{a \in A}(a, x)$, where $\langle \cdot, \cdot\rangle$ is the inner product. Also for a nonzero $z \in \mathbb{R}^3$ we define a *support set* $A(z) = \{a \in A \,|\, \langle a, z\rangle = h_A(z)\}$. We denote

$$\mathcal{B}_G(\mathbb{R}^3) = \{A \in \mathcal{B}(\mathbb{R}^3) \,|\, A = \{x \in \mathbb{R}^3 \,|\, \langle x, z_i\rangle \leq h_A(z_i), i = 1, ..., m\}\}.$$

Elements of the family $\mathcal{B}_G = \mathcal{B}_G(\mathbb{R}^3)$ will be called *G-polyhedra*.

Notice that unless $0 \in \operatorname{int}\operatorname{conv}G$ the elements of $\mathcal{B}_G$ would be unbounded polyhedral sets. Also if $A \in \mathcal{B}_G$ then all normal vectors of two-dimensional faces (support sets) of A belong to G. An important weakness of $\mathcal{B}_G$ is that usually Minkowski sum $A+B$ of G-polyhedra A and B are not G-polyhedra. That problem does not appear if $m = 4$ or if G is the set of six elements of S^2 contained in axes of coordinates. But consider the set of eight normal vectors to the octahedron which is a unit ball in $\mathbb{R}^3$ with norm $||\cdot||_1$ from l^1. Then segments I, J parallel respectively to x_1-axis and x_2-axis are G-polyhedra. However, $I + J$ is a rectangle parallel to x_1x_2-plane and it is not a G-polyhedron.

We can escape this problem in the following way. Let us fix a G-polyhedron K where the set of normal vectors of K is equal to G. By $\mathcal{B}_K(\mathbb{R}^3)$ we define the family of all convex sets homothetic with some summand of K. Then a Minkowski sum of K-polyhedra is always a K-polyhedron. However, growing crystal has fixed possible planes of their faces (it is always a G-polyhedron) and usualy there is no G-polyhedron K such that crystal is always homotetic with some summand of K. Though every $\mathcal{B}_G(\mathbb{R}^3)$ is contained in some $\mathcal{B}_K(\mathbb{R}^3)$, usually such K has much more two-dimensional faces than m. For example if G is a set of 8 vectors normal to faces of regular octahedron then K is a set of 14 vectors normal to faces of truncated octahedron and if G is a set of 14 vectors normal to faces of truncated octahedron then K is a set of 26 vectors normal to faces of truncated cuboctahedron.

Let us define a generalization of Minkowski duality to the family $\mathcal{B}_G$. For $A \in \mathcal{B}_G$ let $h^A = \{h_A(z_1), ..., h_A(z_m)\} \in \mathbb{R}^m$. Notice that $h_A : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is a support function or the polyhedron A and that h^A is a vector corresponding to the values of h_A on discrete subset of m elements of the unit sphere S^2. Let $\mathcal{S}_G = \{h^A \,|\, A \in \mathcal{B}_G\} \subset \mathbb{R}^m$. As we mentioned before, for $A, B \in \mathcal{B}_G$ the Minkowski sum $A+B$ usually does not belong to $\mathcal{B}_G$. Therefore, let us define

$$A \overset{G}{+} B = \{x \in \mathbb{R}^3 \,|\, \langle x, z_i\rangle \leq h_i^A + h_i^B, i = 1, ..., m\}.$$

Then

$$A \overset{G}{+} B \in \mathcal{B}_G,$$

and we have

$$h^{A \overset{G}{+} B} = h^A + h^B \in \mathcal{S}_G.$$

Also $h^{tA} = th^A$ for $t \geq 0$. Hence the set $\mathcal{B}_G$ with the addition $\overset{G}{+}$ and multiplication by nonnegative numbers is an abstract convex cone, i.e. $(\mathcal{B}_G, \overset{G}{+})$ is a commutative semigroup with zero such that

$$\begin{array}{lrcl}
\text{(i)} & (A \overset{G}{+} B) \overset{G}{+} C & = & A \overset{G}{+} (B \overset{G}{+} C), \\
\text{(ii)} & A \overset{G}{+} B & = & B \overset{G}{+} A, \\
\text{(iii)} & A \overset{G}{+} \{0\} & = & A,
\end{array}$$

for all $A, B, C \in \mathcal{B}_G$, and also

$$\begin{array}{llcl} \text{(iv)} & 1A & = & A, \\ \text{(v)} & t(sA) & = & (ts)A, \\ \text{(vi)} & (t+s)A & = & tA \overset{G}{+} sA, \\ \text{(vii)} & t(A \overset{G}{+} B) & = & tA \overset{G}{+} tB, \end{array}$$

for all $A, B \in \mathcal{B}_G$ and $s, t \geq 0$.

Also $\mathcal{S}_G$ is a convex cone in $\mathbb{R}^m$.

On the other hand, for $h \in \mathcal{S}_G$ we define a convex polyhedron

$$A_h = \{x \in \mathbb{R}^3 \mid \langle x, z_i \rangle \leq h_i, i = 1, ..., m\} \in \mathcal{B}_G.$$

Obviously,

$$A_{h+g} = A_h \overset{G}{+} A_g \quad \text{and} \quad A_{th} = tA_h$$

for $h, g \in \mathcal{S}_G$ and $t \geq 0$.

Notice that $h^{tA} = th^A$ for negative t if and only if the set A is a singleton. In a similar way also $A_{th} = tA_h$ for negative t if and only if the set A_h is a singleton.

The mappings

$$\mathcal{B}_G \ni A \longmapsto h_A \in \mathcal{S}_G \quad \text{and} \quad \mathcal{S}_G \ni h \longmapsto A_h \in \mathcal{B}_G$$

are mutually inverse, and the abstract convex cones $(\mathcal{B}_G, \overset{G}{+}, \cdot)$ and $(\mathcal{S}_G, +, \cdot)$ are isomorphic. This way any polyhedron in $\mathcal{B}_G$ can be considered a point in $\mathcal{S}_G \subset \mathbb{R}^m$.

The vector h^A represents the polyhedron A in a manner of [3].

3. Family of symmetric convex polyhedra

Crystals grow in a symmetric manner, since certain subsets of their faces grow with equal rates. Therefore, we can consider a subfamily $\mathcal{B}'_G \subset \mathcal{B}_G$ consisting of polyhedra with

$$\begin{aligned} h_A(z_1) = ... = h_A(z_{k_1}), h_A(z_{k_1+1}) = ... = h_A(z_{k_2}), ..., h_A(z_{k_{l-1}+1}) \\ = ... = h_A(z_{k_l=m}). \end{aligned}$$

The corresponding cone $\mathcal{S}'_G \subset \mathcal{S}_G$ can be looked upon as a subset of $\mathbb{R}^l$. Namely, for $A \in \mathcal{B}'_G$ we can define

$$h'^A = (h_A(z_{k_1}), ..., h_A(z_{k_l})).$$

Then

$$A = \{x \mid \langle x, z_i \rangle \leq h_A(z_{k_j}) \text{ for } k_{j-1} < i \leq k_j\}.$$

Example 3.1. A family $\mathcal{B}'_G$ of polyhedra represented by a three dimensional cone $\mathcal{S}'_G$. Let G be the set of 26 vectors normal to the faces of truncated cuboctahedron i.e.

$$z_1 = (1,0,0),\ z_2(-1,0,0),\ z_3 = (0,1,0), ..., z_7 = \frac{1}{\sqrt{2}}(1,1,0),\ z_7 = \frac{1}{\sqrt{2}}(1,1,0),$$

$$z_8 = \frac{1}{\sqrt{2}}(1,-1,0), ..., z_{19} = \frac{1}{\sqrt{3}}(1,1,1), ..., z_{26} = \frac{1}{\sqrt{3}}(-1,-1,-1).$$

Vectors z_{1-6} correspond to the 6 faces of a cube, vectors z_{7-18} correspond to the 12 edges of a cube and vectors z_{19-26} to the 8 vertices. We consider a subfamily of $\mathcal{B}'_G \subset \mathcal{B}_G$ consisting of polyhedra with

$$h_A(z_1) = ... = h_A(z_6),\ h_A(z_7) = ... = h_A(z_{18}),\ h_A(z_{19}) = ... = h_A(z_{26}).$$

Polyhedra from $\mathcal{B}'_G$ represent many crystals, e.g. crystals of NaCl. Now, for $A \in \mathcal{B}'_G$ we have $h'^A = (h_A(z_6), h_A(z_{18}), h_A(z_{26}))$. Let us notice that $h' = (h'_1, h'_2, h'_3) \in \mathcal{S}'_G$ if and only if the following system of inequalities is satisfied:

$$\begin{cases} h'_1 \leq \sqrt{2}h'_2, \\ h'_2 \leq \sqrt{2}h'_1, \\ h'_2 \leq \frac{\sqrt{6}}{2}h'_3, \\ h'_3 \leq \frac{\sqrt{6}}{2}h'_2. \end{cases}$$

Let us notice that $\mathcal{S}'_G$ is a three-dimensional cone, and a section of this cone with the plane $h'_2 = 1$ is a rectangle $[2^{-1/2}, 2^{1/2}] \times [1.5^{-1/2}, 1.5^{1/2}]$. The following figure illustrates this section of $\mathcal{S}'_G$.

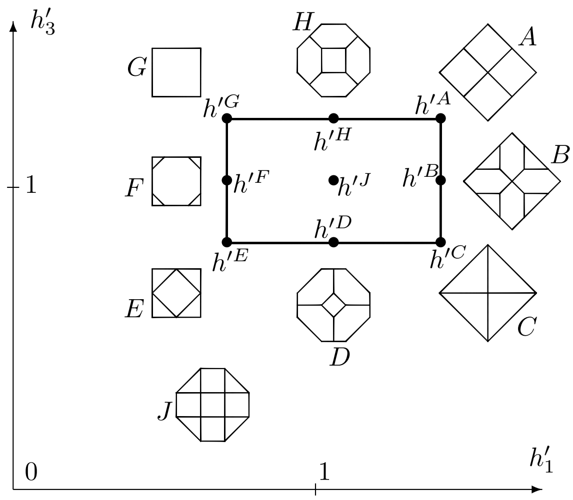

Fig. 3.1: Section of the cone $\mathcal{S}'_G$ with the plane $h'_2 = 0$ in Example 3.1.

Large dots in Fig. 3.1 are points of the cone $\mathcal{S}'_G$ representing certain polyhedra from $\mathcal{B}'_G$. In particular, the vertices of the rectangle represent a rhombidodecahedron A, a regular octahedron C, a cuboctahedron E and a cube G. Side centers of the rectangle represent truncated rhombidodecahedra

$$B = \frac{1}{2}(A \overset{G}{+} C) \quad \text{and} \quad H = \frac{1}{2}(A \overset{G}{+} G),$$

a truncated octahedron

$$D = \frac{1}{2}(C \overset{G}{+} E)$$

and a truncated cube

$$F = \frac{1}{2}(E \overset{G}{+} G).$$

In all these cases the sum $\overset{G}{+}$ coincides with the usual Minkowski sum $+$. The center of the rectangle represents a (small) rhombicuboctahedron

$$J = \frac{1}{2}(A \overset{G}{+} E) = \frac{1}{2}(C \overset{G}{+} G) = \frac{1}{2}(B \overset{G}{+} E) = \frac{1}{2}(D \overset{G}{+} H).$$

The following figure shows that only the sum $C \overset{G}{+} G$ coincides with $C + G$. Other sums differ.

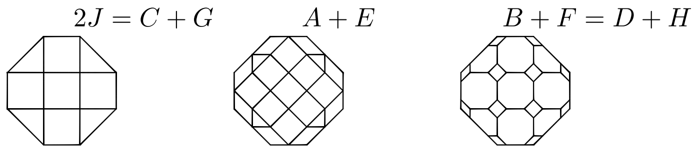

Fig. 3.2: Minkowski sums $A + E, B + F, C + G$ and $D + H$ from Example 3.1.

In the following example we show a trajectory of a growing crystal within the cone described in Example 3.1.

Example 3.2. Growing crystal of NaCl. Let $A(t), t \geq 0$ be a certain continuum of crystals (polyhedra) from the family $\mathcal{S}'_G$ from Example 3.1. A variable t represents time. We assume that the seed $A(0)$ of a crystal is defined by a system of inequalities

$$\begin{cases} \langle x, z_1 \rangle \leq 1, \\ \qquad \vdots \\ \langle x, z_{26} \rangle \leq 1, \end{cases}$$ and that the faces of our crystal parallel to the faces of a cube

grow with a constant speed 1, the faces corresponding to the edges of a cube grow with a constant speed 2 and the faces corresponding to the vertices of a cube grow with a constant speed 3. The changing shape of the crystal is illustrated in Fig. 3.3.

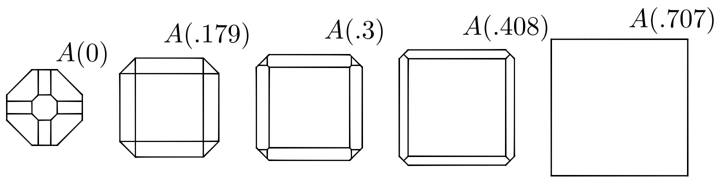

Fig. 3.3: Growing crystal of NaCl from Example 3.2.

Notice that

$$h^{A(t)} = (\overbrace{1+t,...,1+t}^{6}, \overbrace{1+2t,...,1+2t}^{12}, \overbrace{1+3t,...,1+3t}^{8})$$

and

$$h'^{A(t)} = (1+t, 1+2t, 1+3t) \quad \text{for} \quad t \leq 6^{-1/2} \approx .408.$$

For $t \in [6^{-1/2}, 2^{-1/2}]$ we have

$$h'^{A(t)} = (1+t, 1+2t, 1.5^{1/2} + 6^{1/2}t).$$

In this interval faces of $A(t)$ corresponding to the vertices of the cube disappear. At last for $t \geq 2^{-1/2} \approx .707$ we have

$$h'^{A(t)} = (1+t, 2^{1/2} + 2^{1/2}t, 3^{1/2} + 3^{1/2}t) \quad \text{and} \quad A(t)$$

is a growing cube. Fig. 3.4 shows the trajectory of the point $h'^{A(t)}$ corresponding to the growing crystal $A(t)$ within the cone $\mathcal{B}'_G$.

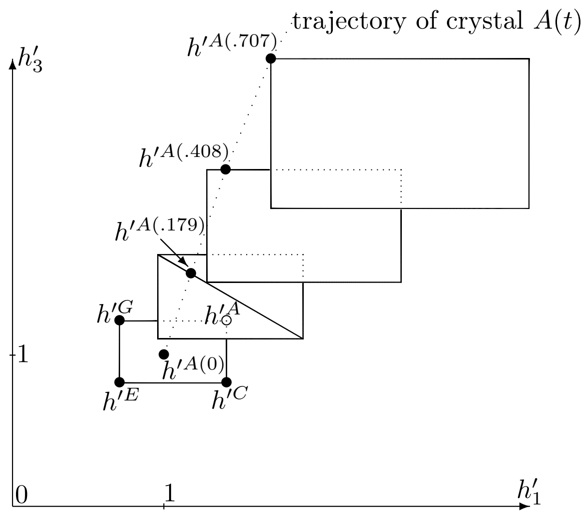

Fig. 3.4. Sections of the cone $\mathcal{S}'_G$ in Example 3.2.

Four rectangles in Fig. 3.4 represent intersections of the cone $\mathcal{B}'_G$ with planes parallel to the plane $0h'_1h'_3$ and containing the points $h'^{A(0)}$, $h'^{A(.179)}$, $h'^{A(.408)}$ and $h'^{A(.707)}$. The trajectory of $h'^{A(t)}$ is a broken line consisting of two segments $[h'^{A(0)}, h'^{A(.408)}]$ and $[h'^{A(.408)}, h'^{A(.707)}]$, and a ray. The first segment is contained in the interior of the cone $\mathcal{B}'_G$ and crosses the diagonal of the rectangle in the point $h'^{A(.179)}$. The crossing point represents the time when some edges of the crystal $A(t)$ disappear and some other ones appear. In other words the faces of the crystal change the number of sides. In particular octagonal faces become squares, rectangular faces become octagonal and hexagonal faces become triangles. The second segment of the trajectory is contained in the relative interior of the face of the cone $\mathcal{B}'_G$. Lastly the ray of the trajectory is contained in the unbounded edge of the cone $\mathcal{B}'_G$.

4. Minkowski difference of polyhedra from the cone $\mathcal{B}_G$

The Minkowski difference of compact convex sets A and B is defined in the following way:

$$A \dot{-} B = \{x \in X \mid x + B \subset A\}.$$

Let us observe that nonempty difference $A \dot{-} B$ of $A, B \in \mathcal{B}_G$ also belongs to $\mathcal{B}_G$. Also $(A \dot{-} B) \overset{G}{+} B \subset A$.

The cone $\mathcal{B}(\mathbb{R}^3)$ can be embeded (in a manner well described in [2]) into a vector space $\mathrm{MRH}(\mathbb{R}^3)$ of virtual bodies, which is a quotient space $\mathcal{B}(\mathbb{R}^3)^2/_{\sim}$, where $(A, B) \sim (C, D)$ if and only if $A + D = B + C$, and $[A, B]$ is a quotient class of (A, B). In a similar way the cone $\mathcal{B}_G$ can be embeded into a vector space $\mathrm{MRH}(\mathcal{B}_G)$ of virtual polyhedra, which is a quotient space $\mathcal{B}_G^2/_{\approx}$, where $(A, B) \approx (C, D)$ if and only if

$$A \overset{G}{+} D = B \overset{G}{+} C, \quad \text{and} \quad [A, B]_G$$

is a quotient class of (A, B). Let us notice that for $A, B, C, D \in \mathcal{B}_G$ the relation $(A, B) \sim (C, D)$ implies $(A, B) \approx (C, D)$ but not conversly. E.g. in Example 3.1 we have

$$A + E \neq C + G \quad \text{but} \quad A \overset{G}{+} E = C \overset{G}{+} G.$$

Then $(C, E) \approx (A, G)$, or $(C, E) \in [A, G]_G$ but $(C, E) \notin [A, G]$.

In the following example we apply the formula

$$A(u) = \frac{u - s}{t - s} A(t) \dot{-} \frac{u - t}{t - s} A(s), \quad s < t < u$$

from [1] to the growing crystal from Example 3.1.

Example 4.1. Let $p([A, B]) = A \dot{-} B$ be a function (projection) from the subset of $\mathrm{MRH}(\mathcal{B}_G)$ to $\mathcal{B}_G$. Let us notice that the definition of p does not depend on the choice of element of the quotient class $[A, B]$. Also $p([A, \{0\}]) = A$ for all $A \in \mathcal{B}_G$. Applying the formula from [1] to the growing crystal $A(t)$ from Example 3.1 for $u > .1$ we obtain

$$\begin{aligned} A(u) &= \frac{u - 0}{0.1 - 0} A(0.1) \dot{-} \frac{u - 0.1}{0.1 - 0} A(0) = 10u A(0.1) \dot{-} (10u - 1) A(0) \\ &= p([10u A(0.1), (10u - 1) A(0)]) = p([A(0.1), \{0\}] + (10u - 1)[A(0.1), A(0)]) \\ &= p([A(0.1), \{0\}] + (u - 0.1)[C, D]), \end{aligned}$$

where C is a cube defined by

$$h'^C = (3\sqrt{3} - 2\sqrt{2}, 3\sqrt{6} - 4, 9 - 2\sqrt{6})$$

and D is a truncated octahedron given by

$$h'^D = (3\sqrt{3} - 2\sqrt{2} - 1, 3\sqrt{6} - 6, 6 - 2\sqrt{6}).$$

Let us notice that the element $[C, D]$ of the space $\mathrm{MRH}(\mathcal{B}_G)$ can be understood as a constant velocity vector with its origin in D and its end in C. We found the polyhedra C and D such that $(C, D) \in 10[A(0.1), A(0)]$ and (C, D) is a minimal pair of sets (or minimal element of $10[A(0.1), A(0)]$). In fact (C, D) is a minimal element of $10[A(0.1), A(0)]_G$, because the origin h'^D of the vector $\overrightarrow{h'^D h'^C}$ is contained in a face of the cone $\mathcal{S}'_G$ and the endpoint h'^C is contained in an extreme ray of the cone $\mathcal{S}'_G$. Figuratively speaking the vector $\overrightarrow{h'^D h'^C}$ cannot be pushed deeper in the cone $\mathcal{S}'_G$.

The projection p is coresponding to the projection $p' : \mathbb{R}^3_+ \longrightarrow \mathcal{S}'_G$ in such a way that

$$p'(h'^A - h'^B) = h'^{(A \dot{-} B)}.$$

In the case of Example 3.1 we can give an explicit formula of p'. Namely,

$$p'(h') = g'$$

if and only if

$$g'_1 = \min(h'_1, \sqrt{2}h'_2, \sqrt{3}h'_3),$$

$$g'_2 = \min(\sqrt{2}h'_1, h'_2, \frac{\sqrt{3}}{\sqrt{2}}h'_3),$$

$$g'_3 = \min(\sqrt{3}h'_1, \frac{\sqrt{3}}{\sqrt{2}}h'_2, h'_3).$$

The formula implies that

$$\text{(i) if } \begin{cases} g'_1 < \sqrt{2}g'_2, \\ g'_2 < \sqrt{2}g'_1, \\ g'_2 < \frac{\sqrt{6}}{2}g'_3, \\ g'_3 < \frac{\sqrt{6}}{2}g'_2, \end{cases} \quad \text{then } p'(h') = g' \text{ only for } h' = g',$$

$$\text{(ii) if } \begin{cases} g'_1 < \sqrt{2}g'_2, \\ g'_2 < \sqrt{2}g'_1, \\ g'_3 = \frac{\sqrt{6}}{2}g'_2, \end{cases} \quad \text{then } p'(h') = g' \text{ for } \begin{cases} h'_1 = g'_1, \\ h'_2 = g'_2, \\ h'_3 \geq g'_3, \end{cases}$$

$$\text{(iii) if } \begin{cases} g'_1 = \sqrt{2}g'_2, \\ g'_2 < \frac{\sqrt{6}}{2}g'_3, \\ g'_3 < \frac{\sqrt{6}}{2}g'_2, \end{cases} \quad \text{then } p'(h') = g' \text{ for } \begin{cases} h'_1 \geq g'_1, \\ h'_2 = g'_2, \\ h'_3 = g'_3, \end{cases}$$

(iv) if $\begin{cases} g_1' < \sqrt{2}g_2', \\ g_2' \le \sqrt{2}g_1', \\ g_2' = \frac{\sqrt{6}}{2}g_3', \end{cases}$ or $\begin{cases} g_2' = \sqrt{2}g_1', \\ g_2' < \frac{\sqrt{6}}{2}g_3', \\ g_3' < \frac{\sqrt{6}}{2}g_2', \end{cases}$ then $p'(h') = g'$ for $\begin{cases} h_1' = g_1', \\ h_2' \ge g_2', \\ h_3' = g_3', \end{cases}$

(v) if $\begin{cases} g_2' = \sqrt{2}g_1', \\ g_2' > 0, \\ g_3' = \frac{\sqrt{6}}{2}g_2', \end{cases}$ then $p'(h') = g'$ for $\begin{cases} h_1' = g_1', \\ h_2' \ge g_2', \\ h_3' \ge g_3', \end{cases}$

(vi) if $\begin{cases} g_1' = \sqrt{2}g_2', \\ g_2' > 0, \\ g_3' = \frac{\sqrt{6}}{2}g_2', \end{cases}$ then $p'(h') = g'$ for $\begin{cases} h_1' \ge g_1', \\ h_2' = g_2', \\ h_3' \ge g_3', \end{cases}$

(vii) if $\begin{cases} g_1' = \sqrt{2}g_2', \\ g_2' > 0, \\ g_2' = \frac{\sqrt{6}}{2}g_3', \end{cases}$ then $p'(h') = g'$ for $\begin{cases} h_1' \ge g_1', \\ h_2' \ge g_2', \\ h_3' = g_3', \end{cases}$ and

(viii) if $g' = 0$ then $p'(h') = g'$ for $h' \ge 0$ such that $h_1'h_2'h_3' = 0$.

5. Conclusions

In [3] authors study polyhedra representing certain crystals restricting themselves to the family of polyhedra homothetic to summands of one fixed polyhedron. In this paper we set a goal to drop the restriction of "homothetic to summands of one fixed polyhedron" in favor of "with faces parallel to fixed set of planes". In [3] the authors do not need to modify Minkowski addition. However, our approach is adequate to the physical nature of crystal growth.

Our approach will enable us to represent any G-polyhedron as Minkowski sum of undecomposable bodies. We also hope to find a way to construct minimal pairs of G-polyhedra (minimal representation of vectors with the origin and the endpoint in the cone of G-polyhedra).

References

[1] J. Grzybowski and R. Urbański, *Crystal growth in terms of Minkowski–Rådström–Hörmander space*, Bull. Soc. Sci. Lettres Łódź Sér. Rech. Déform. **59**, no. 1 (2009), 91–101.

[2] D. Pallaschke and R. Urbański, *Pairs of Compact Convex Sets, Fractional Arithmetic with Convex Sets*, Mathematics and Its Applications, Kluwer Academic. Publisher, Dortrecht–Boston–London 2002.

[3] A. Reinhold and H. Briesen, *Convex geometry for the morphological modeling and characterization of crystal shapes*, Part. Part. Syst. Charact. **28** (2011), 37–56.

[4] R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory*, Encyclopedia of Mathematics and Its Applications, Cambridge University Press, Cambridge 2014.

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MODELOWANIE WZROSTU KRYSZTAŁÓW: WIELOŚCIANY WYPUKŁE O WEKTORACH NORMALNYCH Z PEWNEGO USTALONEGO SKOŃCZONEGO ZBIORU WEKTORÓW

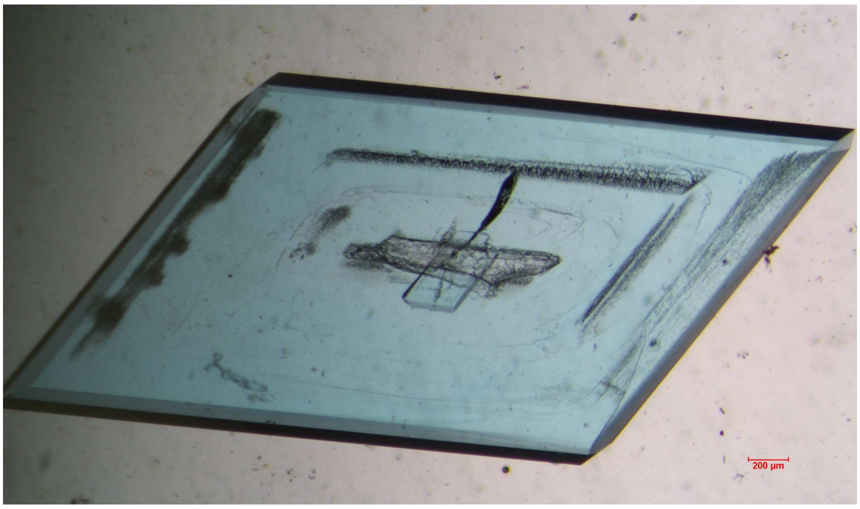
S t r e s z c z e n i e

W artykule przedstawiamy rosnący kryształ jako punkt (wektor) skończenie wymiarowej przestrzeni $\mathbb{R}^m$ poruszający się w stożku $\mathcal{S}_G$ zawartym w $\mathbb{R}^m$. Stożek $\mathcal{S}_G$ odpowiada rodzinie wszystkich wielościanów wypukłych o wektorach normalnych należących do pewnego ustalonego skończonego zbioru wektorów G, czyli G-wielościanów.

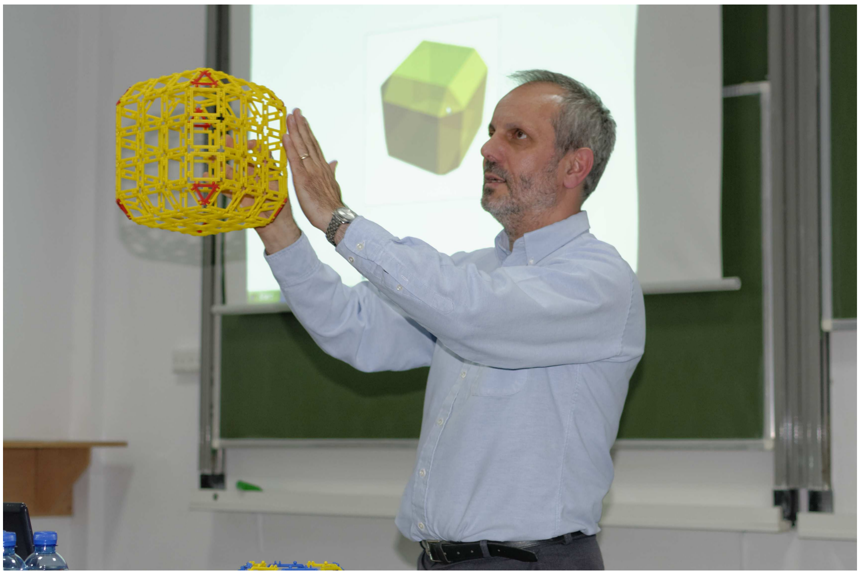
W tym celu modyfikujemy sumę Minkowskiego zbiorów i dogłębnie wyjaśniamy reprezentację wzrostu pewnego modelowego kryształu NaCl. Załączamy również kilka ilustracji odpowiedniego stożka $\mathcal{S}'_G$ symetrycznych wielościanów oraz trajektorię naszego modelowego kryształu, która jest łamaną.

W pracy [3] autorzy badają wielościany przedstawiające pewne kryształy ograniczając się do rodziny wielościanów homotetycznych ze składnikami pewnego ustalonego wielościanu. W przeciwieństwie, porzuciliśmy ograniczenie bycia składnikiem ustalonego wielościanu na rzecz wektorów normalnych z ustalonego zbioru skończonego. Takie podejście wymusiło na nas modyfikację samej sumy Minkowskiego. Jednak wydaje się, że nasze podejście lepiej odzwierciedla fizyczną naturę wzrostu kryształu. Ponadto nasze podejście umożliwia przedstawienie każdego G-wielościanu jako sumy Minkowskiego wielościanów nierozkładalnych. Zamierzamy w przyszłości znaleźć metodę konstruowania par minimalnych G-wielościanów, czyli minimalnej reprezentacji wektorów o początku i końcu należących do stożka G-wielościanów.

Słowa kluczowe: wzrost kryształu, stożek abstrakcyjny wielościanów wypukłych, zmodyfikowana suma Minkowskiego, trajektoria rosnącego kryształu



Photos 1–2: Growth of cooper (II) sulfate hydrate crystal $CuSO_4$
(Marcin Kowiel, Chair of Organic Chemistry, Medical University in Poznań).



Photos 3–4: Models of a crystal of NaCl.

Photos 5–6: String of summands of truncated cubooctahedron and a few other Johnson solids.

Photos 7–8: String of summands of truncated cubooctahedron and a few other Johnson solids (continued).

Photos 9–10: String of summands of truncated cubooctahedron and a few other Johnson solids (continued).

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Marek Aleksiejczyk

MAGIC MATRICES WITH THEIR POWERS, NUMERICAL RANGES AND SOME OPEN PROBLEMS

Summary

The aim of this paper is to derive new properties of magic matrices, i.e. square matrices of natural entries with equal sums in each row, column, main diagonal and antidiagonal. We focus our attention on squares, higher powers and numerical ranges of magic matrices also on their powers. Some open problems are also proposed.

Keywords and phrases: magic matrices, numerical range

1. Introduction and notations

The aim of this paper is to investigate certain properties of *magic matrices*, i.e. square matrices (of order n) satisfying the following conditions:

(m1) the entries a_{ij} of A belong to the set $\{1, 2, \ldots, n^2\}$,

(m2) if $(i, j) \neq (k, l)$ then $a_{ij} \neq a_{kl}$,

(m3) the sums in each row, each column, main diagonal and antidiagonal are equal.

Some authors by "magic matrix" mean a matrix satisfying only conditions (m2) and (m3) and consider matrices with entries being prime numbers, squares of natural numbers, etc. In this paper we call such mnatrices *semi-magic*. Magic matrices (magic squares) were already known 4000 years ago and they still remain an interesting area of research and education [3].

The MATLAB package is a basic software for numerical linear algebra and it is equipped with a build-in function magic(n) which returns a magic matrix of order n.

We will use the following notations throughout this paper:

$M_n(\mathbf{C})$ – the algebra of all complex matrices of order n;

$\|A\| := \sup\{\|Ax\|; \|x\| = 1\}$ – the operator norm of $A \in M_n(\mathbf{C})$;

$\|A\|_1 := \max_j \sum_{i=1}^n |a_{ij}|$ – the 1-norm of $A \in M_n(\mathbf{C})$;

$\|A\|_\infty := \max_i \sum_{j=1}^n |a_{ij}|$ – the infinity-norm of $A \in M_n(\mathbf{C})$;

$\|A\|_F := (\sum_{i,j=1}^n |a_{ij}|^2)^{1/2}$ – the Frobenius (Euclidean) norm of $A \in M_n(\mathbf{C})$;

$\sigma(A)$ – the spectrum of $A \in M_n(\mathbf{C})$;

A^T, A^* – the transpose and adjoint of $A \in M_n(\mathbf{C})$ respectively;

$\operatorname{tr} A := \sum_{i=1}^n a_{ii}$ – the trace of $A = (a_{ij})_{i,j=1}^n \in M_n(\mathbf{C})$;

$\operatorname{conv} D$ – the convex hull of the set $D \subset \mathbf{C}$;

$A \oplus B$ – the direct sum of the matrices A and B.

Let $A \in M_n(\mathbf{C})$ be a magic matrix. It is trivial to notice that the sum determined by condition (m3) is equal to

$$s = \frac{n(n^2+1)}{2}.$$

From now on s will denote this number. We easily verify that s is an eigenvalue of the magic matrix and a corresponding eigenvector is $x = (1, 1, \ldots, 1)^T$. It may be difficult to compute other eigenvalues (see for example [10]); however, in view of the Perron-Frobenius theorem [5] their moduli must be strictly less than s. Let $B \in M_n(\mathbf{R})$ be a positive matrix, let $\lambda_1 > 0$ denote its eigenvalue with largest modulus and let m and M denote its smallest and largest entry. The following estimate of Hopf [12] holds:

$$|\lambda_i| \le \lambda_1 \frac{M-m}{M+m} \qquad (i = 2, \ldots, n),$$

optimal in the general case. In the further part of this paper we give a better estimate for magic matrices. Since s is an eigenvalue of A and $\|A\|^2 \le \|A\|_1 \|A\|_\infty$ we have $\|A\| = s$.

Let us now recall the celebrated Schur theorem (1909):

Theorem 1. *Every matrix $A \in M_n(\mathbf{C})$ is unitarily similar to an upper triangular matrix, i.e. $A = UTU^*$, where $U^*U = I$, and $T = (t_{ij})$ satisfies $t_{ij} = 0$ if $j > i$.*

The original proof was published in [15] but it may also be found in many textbooks (see for example [14]).

The following famous theorem was proved by Gerschgorin in 1931 [6]:

Theorem 2. *The spectrum of $A \in M_n(\mathbf{C})$ is contained in the union $G(A)$ of the discs*

$$G_i(A) := \Big\{ z \in \mathbf{C} : |a_{ii} - z| \le \sum_{j \ne i} |a_{ij}| \Big\} \qquad (i = 1, \ldots, n).$$

Moreover, if k of the Gerschgorin discs form a connected set, then this set contains at precisely k eigenvalues (counted with multiplicities).

For the proof see also [11, 14].

Remark 1. *A. Zalewska-Mitura and J. Zemánek considered in* [16] *the set*

$$\mathcal{Z}(A) := \bigcap \{G(U^*AU) : U \text{ unitary}\}$$

and proved that for some matrices $\mathcal{Z}(A) = \sigma(A)$ (for example matrices whose squares are normal, and matrices satisfying quadratic equation) but this equality is not valid in general. Some open problems were also proposed.

The next theorem was proved by T. J. Laffey (1996) (see [16]).

Theorem 3. *Let $A \in M_n(\mathbf{C})$ be such that A^2 is normal. Then A is unitarily similar to a direct sum of blocks of dimension at most 2.*

2. Properties of certain magic matrices and their powers

In this section we discuss some properties of all 3×3 magic matrices and of magic matrices of orders $4k$, $k = 1, 2, \ldots$, generated by MATLAB and their powers.

At first let us consider magic matrices of order 3. It is known and easy to verify by computer that there are only eight magic matrices of order 3:

(1)

$$A_1 = \begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix}, A_2 = \begin{bmatrix} 8 & 3 & 4 \\ 1 & 5 & 9 \\ 6 & 7 & 2 \end{bmatrix}, A_3 = \begin{bmatrix} 2 & 9 & 4 \\ 7 & 5 & 3 \\ 6 & 1 & 8 \end{bmatrix}, A_4 = \begin{bmatrix} 2 & 7 & 6 \\ 9 & 5 & 1 \\ 4 & 3 & 8 \end{bmatrix},$$

(2)

$$A_5 = \begin{bmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{bmatrix}, A_6 = \begin{bmatrix} 4 & 3 & 8 \\ 9 & 5 & 1 \\ 2 & 7 & 6 \end{bmatrix}, A_7 = \begin{bmatrix} 6 & 1 & 8 \\ 7 & 5 & 3 \\ 2 & 9 & 4 \end{bmatrix}, A_8 = \begin{bmatrix} 6 & 7 & 2 \\ 1 & 5 & 9 \\ 8 & 3 & 4 \end{bmatrix}.$$

Because all matrices of the form (1) may be obtained from A_1 by transpositions and unitary similarities, and A_5 plays the same role for (2), it is enough to find formulas for the powers of A_1 and A_5 to obtain formulas for the powers of all magic matrices of order 3.

The following may be easily verified by induction:

Lemma 1. *Let $a := 225$, $b := 24$. Then, for $k = 0, 1, 2, \ldots$,*

$$A_1^{2k+1} = \begin{bmatrix} 5a^k + 3b^k & 5a^k - 4b^k & 5a^k + b^k \\ 5a^k - 2b^k & 5a^k & 5a^k + 2b^k \\ 5a^k - b^k & 5a^k + 4b^k & 5a^k - 3b^k \end{bmatrix},$$

$$A_1^{2k+2} = \begin{bmatrix} 75a^k + 16b^k & 75a^k - 8b^k & 75a^k - 8b^k \\ 75a^k - 8b^k & 75a^k + 16b^k & 75a^k - 8b^k \\ 75a^k - 8b^k & 75a^k - 8b^k & 75a^k + 16b^k \end{bmatrix},$$

$$A_5^{2k+1} = \begin{bmatrix} 5a^k + (-1)^{k+1} \cdot b^k & 5a^k - (-1)^{k+1} \cdot 4b^k & 5a^k + (-1)^{k+1} \cdot 3b^k \\ 5a^k + (-1)^{k+1} \cdot 2b^k & 5a^k & 5a^k - (-1)^{k+1} \cdot 2b^k \\ 5a^k - (-1)^{k+1} \cdot 3b^k & 5a^k + (-1)^{k+1} \cdot 4b^k & 5a^k - (-1)^{k+1} \cdot b^k \end{bmatrix},$$

$$A_5^{2k+2} = \begin{bmatrix} 75a^k + (-1)^{k+1} \cdot 16b^k & 75a^k - (-1)^{k+1} \cdot 8b^k & 75a^k - (-1)^{k+1} \cdot 8b^k \\ 75a^k - (-1)^{k+1} \cdot 8b^k & 75a^k + (-1)^{k+1} \cdot 16b^k & 75a^k - (-1)^{k+1} \cdot 8b^k \\ 75a^k - (-1)^{k+1} \cdot 8b^k & 75a^k - (-1)^{k+1} \cdot 8b^k & 75a^k + (-1)^{k+1} \cdot 16b^k \end{bmatrix}.$$

We easily obtain two corollaries:

Corollary 1.

$$A_1^{4n} = A_5^{4n} \qquad \textit{for} \qquad n = 1, 2, 3, \ldots.$$

Corollary 2. *Odd powers of magic matrices of order 3 are semi-magic matrices, and even powers are symmetric Toeplitz matrices* (*and multiples of doubly stochastic matrices*)*.*

Next we turn our attention to magic matrices of order $4k$, $k = 1, 2, \ldots$, generated by MATLAB. The following result was proved by Kirkland and Neumann in [10]:

Proposition 1. *The magic matrix* $A = \mathrm{magic}(n)$*, where* $n \equiv 0 \mod 4$*, generated by MATLAB, given by*

$$a_{i,j} = \begin{cases} (i-1)n + j & \textit{if} \begin{cases} i \equiv 0,1 \mod 4 \textit{ and } j \equiv 2,3 \mod 4 \\ \textit{or} \\ i \equiv 2,3 \mod 4 \textit{ and } j \equiv 0,1 \mod 4 \end{cases} \\ n^2 - (i-1)n - j + 1 & \textit{if} \begin{cases} i \equiv 0,1 \mod 4 \textit{ and } j \equiv 0,1 \mod 4 \\ \textit{or} \\ i \equiv 2,3 \mod 4 \textit{ and } j \equiv 2,3 \mod 4 \end{cases} \end{cases}$$

is of rank three and its nonzero eigenvalues are

$$s = \frac{n(n^2+1)}{2},$$

and $\pm a$ *where*

$$a = \frac{n}{2\sqrt{3}}\sqrt{n^3 - n}.$$

Now we are able to prove the following lemma:

Lemma 2. *Let* A *be an* $n \times n$ *magic matrix, where* $n \equiv 0 \mod 4$*, generated by MATLAB, and let* $C = A^2$*. Then* C *is symmetric and*

$$c_{i,j} = \begin{cases} \dfrac{3n^5 - 4n^4 + 6(i+j)n^3 - 12ijn^2 + 6(i+j)n^2 - 2n^2 + 3n}{12} \\ \textit{if} \begin{cases} i \equiv 0,1 \mod 4 \textit{ and } j \equiv 2,3 \mod 4 \\ \textit{or} \\ i \equiv 2,3 \mod 4 \textit{ and } j \equiv 0,1 \mod 4 \end{cases} \\ \dfrac{3n^5 + 4n^4 - 6(i+j)n^3 + 12n^3 + 12ijn^2 - 6(i+j)n^2 + 2n^2 + 3n}{12} \\ \textit{if} \begin{cases} i \equiv 0,1 \mod 4 \textit{ and } j \equiv 0,1 \mod 4 \\ \textit{or} \\ i \equiv 2,3 \mod 4 \textit{ and } j \equiv 2,3 \mod 4. \end{cases} \end{cases}.$$

Proof. We consider four cases:
1. $i \equiv 0,1 \mod 4$ and $j \equiv 2,3 \mod 4$
2. $i \equiv 2,3 \mod 4$ and $j \equiv 1,2 \mod 4$
3. $i, j \equiv 0,1 \mod 4$
4. $i, j \equiv 2,3 \mod 4$.

We will prove case 1 for example. Let i, j be such that $i \equiv 0,1 \mod 4$ and $j \equiv 2,3 \mod 4$ and let $m = n/4$, where $n = 4, 8, 12, \ldots$. Then

$$\begin{aligned} c_{ij} = \sum_{k=1}^{n} a_{ik}a_{kj} &= \sum_{k=0}^{m-1} a_{i(4k+1)}a_{(4k+1)j} + \sum_{k=0}^{m-1} a_{i(4k+2)}a_{(4k+2)j} \\ &+ \sum_{k=0}^{m-1} a_{i(4k+3)}a_{(4k+3)j} + \sum_{k=0}^{m-1} a_{i(4k+4)}a_{(4k+4)j} \end{aligned}$$

$$= \sum_{k=0}^{m-1} (a_{i(4k+1)}a_{(4k+1)j} + a_{i(4k+2)}a_{(4k+2)j} + a_{i(4k+3)}a_{(4k+3)j} + a_{i(4k+4)}a_{(4k+4)j})$$

$$\begin{aligned} &= \sum_{k=0}^{m-1} ([(16m^2 - (i-1)4m - 4k)(16km + j)] \\ &+[((i-1)4m + 4k + 2)(16m^2 - (4k+1)4m - j + 1)] \\ &+[((i-1)4m + 4k + 3)(16m^2 - (4k+2)4m - j + 1)] \\ &+[(16m^2 - (i-1)4m - 4k - 3)((4k+3)4m + j)]) \end{aligned}$$

$$\begin{aligned} = \sum_{k=0}^{m-1} &([-64mk^2 + (256m^3 - 64im^2 + 64m^2 - 4j)k + 16jm^2 - 4ijm + 4jm] \\ &+[-64mk^2 + (-64im^2 + 128m^2 - 48m - 4j + 4)k + 64im^3 \\ &-64m^3 - 16im^2 + 48m^2 - 4ijm + 4(i+j)m - 12m - 2j + 2] \\ &+[-64mk^2 + (-64im^2 + 128m^2 - 80m - 4j + 4)k + 64im^3 \\ &-64m^3 - 32im^2 + 80m^2 - 4ijm + 4(i+j)m - 28m - 3j + 3] \end{aligned}$$

$$+[-64mk^2 + (256m^3 - 64im^2 + 64m^2 - 96m - 4j)k$$
$$+192m^3 - 48im^2 + 16jm^2 + 48m^2 - 4ijm + 4jm - 36m - 3j])$$
$$= \sum_{k=0}^{m-1} (-256mk^2 + (512m^3 - 256im^2 + 384m^2 - 224m - 16j + 8k)k$$
$$+128im^3 + 64m^3 - 96im^2 + 32jm^2 + 176m^2 - 16ijm + 8im + 16jm - 76m - 8j + 5).$$

Applying well known formulas:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

we obtain after some calculations:

$$c_{ij} = \frac{768m^5 - 256m^4 + 96(i+j)m^3 - 48ijm^2 + 24(i+j)m^2 - 8m^2 + 3m}{3}.$$

Now we replace m by $n = 4m$ and obtain

$$c_{ij} = \frac{768m^5 - 256m^4 + 96(i+j)m^3 - 48ijm^2 + 24(i+j)m^2 - 8m^2 + 3m}{3}$$
$$= \frac{768(\frac{n}{4})^5 - 256(\frac{n}{4})^4 + 96(i+j)(\frac{n}{4})^3 - 48ij(\frac{n}{4})^2 + 24(i+j)(\frac{n}{4})^2 - 8(\frac{n}{4})^2 + 3(\frac{n}{4})}{3}$$
$$= \frac{\frac{768}{256}n^5 - \frac{256}{64}n^4 + \frac{96}{16}(i+j)n^3 - \frac{48}{4}ijn^2 + \frac{24}{4}(i+j)n^2 - \frac{8}{2}n^3 + 3n}{12}$$
$$= \frac{3n^5 - 4n^4 + 6(i+j)n^3 - 12ijn^2 + 6(i+j)n^2 - 2n^2 + 3n}{12},$$

i.e. our formula. Other cases are proved analogously. □

Remark 2. *The property in Lemma* 2 *seems to be unique for magic matrices generated by MATLAB: by interchanging the second and third columns in the matrix* $A = \mathrm{magic}(4)$ *we obtain the magic matrix*

$$B = \begin{bmatrix} 16 & 3 & 2 & 13 \\ 5 & 10 & 11 & 8 \\ 9 & 6 & 7 & 12 \\ 4 & 15 & 14 & 1 \end{bmatrix}$$

whose square is not symmetric.

Corollary 3. *Let* $A = \mathrm{magic}(n)$*, where* $n = 3$ *or* $n \equiv 0 \mod 4$*, be a magic matrix generated by MATLAB. Then* $Z(A) = \sigma(A)$.

3. The numerical range of magic matrices and their powers

In this section we introduce the notion of the numerical range of a square matrix and consider its properties for magic matrices and their powers. Let S be the Euclidean unit sphere in $\mathbf{C}^n$ and let $\langle \cdot, \cdot \rangle$ denote the standard inner product on $\mathbf{C}^n$. For any $A \in M_n(\mathbf{C})$ we define the set

$$W(A) = \{\langle Ax, x\rangle : x \in S\}$$

to be the *numerical range* of A. The set $W(A)$ is also known as the *field of values* of A. The number

$$r(A) = \sup\{|z| : z \in W(A)\}$$

is called the *numerical radius* of A.
In the following lemma we collect some basic properties of $W(A)$ and $r(A)$:

Lemma 3. *Let $A, B \in M_n(\mathbf{C})$. Then*

(3.1) $W(A)$ is a compact and convex subset of the complex plane.

(3.2) $W(U^*AU) = W(A)$ for every unitary matrix U.

(3.3) $W(\alpha A + \beta I) = \beta + \alpha W(A)$ for $\alpha, \beta \in \mathbf{C}$.

(3.4) $W(A + B) \subset W(A) + W(B)$.

(3.5) $W(A \oplus B) = \mathrm{conv}(W(A) \cup W(B))$.

(3.6) $\sigma(A) \subset W(A)$.

(3.7) $W(A^T) = W(A),\ W(A^*) = \overline{W(A)} := \{\bar{z} : z \in W(A)\}$.

(3.8) If A is normal, then $W(A)$ is the convex hull of the spectrum of A.

(3.9) $W(\frac{A+A^*}{2}) = Re(W(A)),\ W(\frac{A-A^*}{2}) = Im(W(A))$.

(3.10) $W(A)$ is a real line segment if and only if A is Hermitian.

(3.11) If A is a 2×2 matrix with eigenvalues p, q then the numerical range of A is an elliptical disc with eigenvalues of A as foci and the minor axis of length $\sqrt{\mathrm{tr}(A^*A) - |p|^2 - |q|^2}$.

(3.12) r is a norm and $\frac{1}{2}\|A\| \le r(A) \le \|A\|$.

For the proofs of these properties and generalizations of the classical numerical range we refer the reader to [9], [8] and the references given there. The numerical radius is a norm but not a matrix norm, i.e. it is not submultiplicative. Moreover, it may even happen that $r(A^{n+m}) > r(A^n)r(A^m)$ (see for example [2]). The numerical radius however satisfies the following *power inequality*:

$$r(A^n) \le r(A)^n \qquad \text{for all } n \in \mathbf{N},$$

proved by Berger [1] in 1965 (for a proof see also [13] and [7]).
The following theorem is an interesting and natural generalization of (3.8); it was proved by Ky Fan in [4].

Theorem 4. *Let A be a bounded linear operator acting on a complex Hilbert space, let λ, μ be distinct eigenvalues of A, and let x, y be its corresponding eigenvectors:*

$Ax = \lambda x$, $Ay = \mu y$. Then if λ lies on the boundary of the numerical range of A, then x and y are orthogonal.

Let A be a magic matrix. Then s is an eigenvalue of A and $\|A\| = s$; hence, in view of (3.6) and (3.12), s always lies on the boundary of $W(A)$. Thus, using the above theorem, we obtain the following corollary.

Corollary 4. *Let A be a magic matrix of order n. Then A is unitarily similar to a matrix of the form*

$$\begin{bmatrix} s & 0 \\ 0 & K \end{bmatrix}$$

for some $K \in M_{n-1}(\mathbf{C})$.

From now on K will denote the lower block of the above matrix.
It may be difficult to say anything about K except two obvious properties:
(3.13) $\operatorname{tr} K = 0$.
(3.14) $\|K\|_F = \sqrt{\|A\|_F^2 - s^2} = \left(\frac{n^2(n^2+1)(n^2-1)}{12}\right)^{1/2}$.
From (3.5) and Corollary 4 we have:

Corollary 5. *Let A be a magic matrix. Then $W(A^n) = \operatorname{conv}(\{s^n\} \cup W(K^n))$.*

Now we will estimate the numerical range of a magic matrix.

Proposition 2. *Let A be a magic matrix. Then the numerical range of A is included in the convex hull of the set consisting of $\{s\}$ and the circular disc with center at 0 and radius $s/\sqrt{3}$.*

Proof. By Corollary 5 it is sufficient to estimate the numerical range of the matrix K. We have

$$r(K)/s \le \|K\|/s \le \|K\|_F/s = \frac{1}{\sqrt{3}}\sqrt{\frac{n^2-1}{n^2+1}} < \frac{1}{\sqrt{3}},$$

which completes the proof. □

Powers of magic matrices are not symmetric in general, but they have the following interesting property:

Proposition 3. *Let A be a magic matrix. Then the successive powers of A are "getting relatively symmetric" in the sense that*

$$\frac{\|A^n - (A^T)^n\|}{\|A^n\|} \to 0 \qquad \text{as } n \to \infty.$$

Proof. We have

$$\frac{\|A^n-(A^T)^n\|}{\|A^n\|}=\frac{\|A^n-(A^*)^n\|}{\|A^n\|}\le\frac{2r(A^n-(A^*)^n)}{\|A^n\|}=\frac{2r(K^n-(K^*)^n)}{\|A^n\|}$$
$$\le 4\frac{r(K^n)}{\|A^n\|}\le 4\frac{r(K)^n}{\|A^n\|}=4\frac{r(K)^n}{s^n}\le 4\Big(\frac{1}{\sqrt{3}}\Big)^n\to 0. \qquad \square$$

Remark 3. *A. Pokrzywa* (*Institute of Mathematics, Polish Academy of Sciences*) *proved a stronger result* (*personal communication*)*: if A is a square matrix of order n with positive entries and with equal sums in all columns and rows* (*denote this number also by s*) *then*

$$\Big(\frac{1}{s}A\Big)^k\to\frac{1}{n}\begin{bmatrix}1&\dots&1\\ \vdots&&\vdots\\ 1&\dots&1\end{bmatrix}\qquad \textit{as } k\to\infty.$$

Now we turn our attentionback to magic matrices of order 3. It is easy to verify that all matrices of the form (1) have eigenvalues $15, 2\sqrt{6}$, $-2\sqrt{6}$ and all matrices of the form (2) have eigenvalues $15, 2\sqrt{6}\,i, -2\sqrt{6}\,i$. Using this fact and keepingin mind that the Frobenius norm is unitarily invariant we obtain the following lemma which will be useful in the proof of Theorem 5.

Lemma 4. *Let A and B be magic matrices of order 3 of the form* (1) *and* (2) *respectively. Then A is unitarily similar to S and B is unitarily similar to T, where*

$$S=\begin{bmatrix}15&0&0\\0&2\sqrt{6}&z_1\\0&0&-2\sqrt{6}\end{bmatrix},\qquad T=\begin{bmatrix}15&0&0\\0&2\sqrt{6}\,i&z_2\\0&0&-2\sqrt{6}\,i\end{bmatrix},$$

where $|z_1|=|z_2|=\sqrt{\|A\|_F^2-15^5-2(2\sqrt{6})^2}=2\sqrt{3}.$

Now we are able to describe the numerical ranges of magic matrices of order 3.

Theorem 5. *Let A be a magic matrix of order* 3.

1) *If A is of the form* (1) *then the numerical range of A^n is the line segment with endpoints $(2\sqrt{6})^n$ and 15^n if n is even, and the convex hull of the set consisting of $\{15^n\}$ and the elliptical disc with foci $-(2\sqrt{6})^n$ and $(2\sqrt{6})^n$ and with minor axis of length $2\sqrt{3}\cdot(2\sqrt{6})^{n-1}$ if n is odd.*

2) *If A is of the form* (2) *then the numerical range of A^n is the line segment with endpoints $(2\sqrt{6}\,i)^n$ and 15^n if n is even, and the convex hull of the set consisting of $\{15^n\}$ and the elliptical disc with foci $-(2\sqrt{6})^n i$ and $(2\sqrt{6})^n i$ and with minor axis of length $2\sqrt{3}\cdot(2\sqrt{6})^{n-1}$ if n is odd.*

Proof. We will prove the first case of our theorem. In view of Corollary 5 it is enough to find the numerical range of the matrix K. Since the numerical range of a matrix of order 2 depends only upon the modulus of the $(1,2)$ entry, we may assume in view of Lemma 4 that

$$K = \begin{bmatrix} 2\sqrt{6} & 2\sqrt{3} \\ 0 & -2\sqrt{6} \end{bmatrix}.$$

We prove by induction that

$$K^n = \begin{bmatrix} (2\sqrt{6})^n & 0 \\ 0 & (2\sqrt{6})^n \end{bmatrix}$$

if n is even and

$$K^n = \begin{bmatrix} (2\sqrt{6})^n & 2\sqrt{3}(2\sqrt{6})^{n-1} \\ 0 & -(2\sqrt{6})^n \end{bmatrix}$$

if n is odd.

It remains to apply (3.11) to get our assertion. □

Now we prove the following result.

Theorem 6. *Let A be an $n \times n$ magic matrix generated by MATLAB, where $n \equiv 0$ mod 4. Then the numerical range of A^k is the line segment with endpoints a^k and s^k if k is even, and the convex hull of the set consisting of $\{s^k\}$ and the elliptical disc with foci $-a^k$ and a^k and with minor axis of length $a^{k-1}b$ if k is odd, where*

$$a = \frac{n}{2\sqrt{3}}\sqrt{n^3 - n}, \qquad b = \frac{\sqrt{3}}{6}n\sqrt{(n+1)(n-1)^3}.$$

Proof. We will first exhibit the Schur form of the matrix K. Snce A^2 is symmetric (Lemma 2), so is K^2. Therefore, in view of Theorem 3, K is unitarily similar to the direct sum of some blocks $S_1, S_2, \ldots, S_m$, where S_i, $i = 1, 2 \ldots, m$, are 1×1 or 2×2 matrices. Since $\operatorname{rank} K = 2$, only at most two of these blocks, say S_1 and S_2, are nonzero. Moreover, we may assume that S_1 and S_2 are upper triangular. Three cases are possible:

1. $S_1 = \begin{bmatrix} a & b \\ 0 & -a \end{bmatrix}$, $S_2 = 0$ or vice versa,

2. $S_1 = \begin{bmatrix} a & c \\ 0 & 0 \end{bmatrix}$, $S_2 = \begin{bmatrix} -a & 0 \\ 0 & 0 \end{bmatrix}$ or vice versa,

3. $S_1 = \begin{bmatrix} a & c_1 \\ 0 & 0 \end{bmatrix}$, $S_2 = \begin{bmatrix} -a & c_2 \\ 0 & 0 \end{bmatrix}$ or vice versa,

for $a = \frac{n}{2\sqrt{3}}\sqrt{n^3 - n}$ (Proposition 1) and some b, c, c_1, c_2.

But $K = S_1 \oplus S_2 \oplus 0 \oplus \ldots \oplus 0$ and K^2 is symmetric so, only case 1 may happen. Let us assume that S_1 is a nonzero block. Then $W(K) = W(S_1)$ and, in view of (3.11), it is enough to find the modulus of b to compute the numerical range of K.

By a trivial calculation we obtain

$$|b| = \sqrt{\|K\|_F^2 - 2 \cdot |a|^2} = \frac{\sqrt{3}}{6} n \sqrt{(n+1)(n-1)^3}.$$

Since the numerical range of K depends only upon the modulus of b; we may consider only real nonnegative values of b and write b instead of $|b|$.
Finally, we verify that

$$S_1^k = \begin{bmatrix} a^k & 0 \\ 0 & a^k \end{bmatrix}$$

if k is even, and

$$S_1^k = \begin{bmatrix} a^k & a^{k-1}b \\ 0 & a^k \end{bmatrix}$$

if k is odd. It is enough to apply 3.11 and 3.5 to get the assertion. □

4. Open problems

Open problem 1. Describe or estimate the number of magic matrices of order n (for small values of n some results are known).

Open problem 2. Magic matrices of odd order seem to be of full rank, while magic matrices of even order seem to have rank strictly smaller than their order. Is that true in general?

Open problem 3. The magic matrices exhibited in Remark 1.1 and Remark 2.1 have different eigenvalues (apart from s) than matrices of the same order generated by MATLAB. How much may the eigenvalues of different magic matrices of the same order differ? May they have an arbitrary phase? Are they always symmetric with respect to zero (not counting s)?

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References

[1] C. A. Berger, *On the numerical range of powers of an operator*, Abstract no. 625–152, Notices Amer. Math. Soc. **12** (1965), 590.

[2] F. F. Bonsall and J. Duncan, *Numerical ranges*, in: *Studies in functional analysis* (ed. R. G. Bartle), Mathematical Association of America: Studies in Mathematics.

[3] R. Descombes, *Les carrés magiques: histoire, théorie et technique du carré magique*, Librairie Vuibert, Paris 2000.

[4] Ky Fan, *A remark on orthogonality of eigenvalues*, Linear and Multilinear Algebra **23** (1988), 283–284.

[5] F. R. Gantmacher, *Theory of Matrices*, Nauka, Moscow 1988 (in Russian).

[6] S. Gerschgorin, *Über die Abgrenzung der Eigenwerte einer Matrix*, Izv. Akad. Nauk USSR **7** (1931), 749–754.

[7] M. Goldberg and E. Tadmor, *On the numerical radius and its applications*, Linear Algebra Appl. **42** (1982), 263–284.

[8] K. E. Gustafson and D. K. M. Rao, *Numerical Range. The field of values of linear operators and matrices*, Springer-Verlag, New York 1997.

[9] [9] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge 1991.

[10] [10] S. J. Kirkland and M. Neumann, *Group inverses of M-matrices associated with nonnegative matrices having few eigenvalues*, Linear Algebra Appl. **220** (1995), 181–213.

[11] [11] P. Lancaster and M. Tismenetsky, *The Theory of Matrices. Second Edition*, Academic Press, 1985.

[12] A. M. Ostrowski, *Positive matrices and functional analysis*, Recent Advances in Matrix Theory, The University of Wisconsin Press, 1964.

[13] C. Pearcy, *An elementary proof of the power inequality for the numerical radius*, Michigan Math. J. **13** (1966), 289–291.

[14] V. V. Prasolov, *Problems and Theorems in Linear Algebra*, Amer. Math. Soc. (1994).

[15] I. Schur, *Über die charakteristischen Wurzeln einer linearen Substitution mit einer Anwendung auf die Theorie der Integralgleichungen*, Math. Ann. **66** (1909), 488–510.

[16] A. Zalewska-Mitura, J. Zemánek, *The Gerschgorin discs under unitary similarity*, Banach Center Publications **38** (1997), 427–441.

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MACIERZE MAGICZNE – POTĘGI, OBRAZY LICZBOWE I OTWARTE PROBLEMY

S t r e s z c z e n i e

W pracy niniejszej rozważamy własności tzw. macierzy magicznych, tj. macierzy o elementach będących liczbami naturalnymi takimi, że sumy wszystkich wierszy, kulumn oraz obu głównych przekątnych są równe. Zajmujemy się własnościami ich potęg i ich obrazów liczbowych. Na zakończenie prezentujemy kilka otwartych problemów.

Słowa kluczowe: magiczne macierze, obrazy liczbowe

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Rafał Zduńczyk

SIMPLE SYSTEMS AND GENERALIZED TOPOLOGIES

Summary

We introduce the notion of a simple system as a generalization of B. Thomson's local system. We use it to construct a generalized topology. For a filtering system the construction leads to a topology (in the classical sense) in which the underlying set is dense-in-itself. We examine the interplay between local systems and related topologies.

Keywords and phrases: local system, generalized topology, generalized open set

Let X be a nonempty set. For an $A \subset X$ by 2^A we denote the power set of A, i.e. the collection of all subsets of A. A class of collections of subsets of X, $\mathcal{S} = \{S(x)\}_{x \in X}$, will be called a ***simple system*** provided the following conditions hold for any $x \in X$:

S1.1. $\{x\} \notin S(x)$, *S1.2.* $S(x) \neq \varnothing$,

S2. if $S \in S(x)$, then $x \in S$,

S3. if $S_1 \in S(x)$ and $S_1 \subset S_2$, then $S_2 \in S(x)$.

Property 1. *Let $d : 2^X \to 2^X$, $d : A \mapsto d\,A$, be an operator such that*

OP1. $d\,X = X$,

OP2. $x \notin d\,\{x\}$,

OP3. *if* $A_1 \subset A_2 \subset X$, *then* $d\,A_1 \subset d\,A_2$.

The collection $\mathcal{S}^d = \{S^d(x)\}_{x\in X}$ *defined by*

$$S \in S^d(x), \qquad \textit{when} \quad x \in S \cap d\,S, \tag{1}$$

is a simple system in X.

Proof. S1 is a consequence of *OP1* and *OP2*. Obviously, *S3* follows from *OP3*, while *S2* is fulfilled by virtue of (1). □

There are two reasons for introducing d here. One is that d leads naturally to a simple system, the other that it will help us building a counterexample in a quite general case in the sequel.

Let γ be a nonempty collection of subsets of X. We will call γ a ***generalized topology*** (a ***g-topology***) and (X, γ) a ***generalized topological space*** (cf. [2]) provided that

GT1. $\varnothing \in \gamma$,

GT2. if $\mathcal{G} \subset \gamma$, then $\bigcup \mathcal{G} \in \gamma$.

Example 1. At first glance it may seem awkward and artificial to drop the condition on intersections from the original definition of a topology. However, a surprisingly natural example of such a structure comes at hand, i.e. so called β-structure (cf. [6], or *semi-open* sets in [5], [4] and [1]), defined as

$$\mathcal{T}^\beta := \{B \subset X : B \subset \mathcal{T}\operatorname{cl}\mathcal{T}\operatorname{Int} B\},$$

for an arbitrary topology $\mathcal{T}$ on X. $\mathcal{T}^\beta$ is always a generalized topology, and happens to be a classical topology only when $\mathcal{T}$ is extremely disconnected (cf. [6]). Note that equality of β-structures gives an equivalence relation among topologies (so called *semi-correspondence* cf. [4]) and in general does not imply equality of these topologies, contrary to Corollary 1 in [5]. ◇

For $A \subset X$ by γ-gInt A we denote the ***generalized*** γ***-interior*** of A, i.e. $\gamma\text{-gInt}\,A := \bigcup\left(\gamma \cap 2^A\right)$.

Property 2. [cf. [2], Lemma 1.2]. *Assume* $\mathcal{S} = \{S(x)\}_{x\in X}$ *is a simple system in* X. *The collection*

$$\gamma_{\mathcal{S}} := \{G \subset X : G \in S(x),\ \textit{for all}\ x \in G\} \tag{2}$$

is a g-topology in X.

Given $\gamma \neq \{\varnothing\}$, a g-topology in X, define

$$S_0^\gamma(x) := \{S \subset X : x \in \gamma\text{-gInt}\,S\}\,. \tag{3}$$

Clearly, $\mathcal{S}_0^\gamma = \{S_0^\gamma(x)\}_{x\in\bigcup\gamma}$ is a simple system in $\bigcup\gamma$ provided there are no singletons in γ (cf. [2], Lemma 1.3). For else, condition *S1.1* would not be satisfied. It is an easy check that by putting

$$\mathcal{S}' \sqsubset \mathcal{S}'', \qquad \text{when} \qquad S'(x) \subset S''(x), \text{ for all } x \in X, \tag{4}$$

we introduce a partial order in the class of all simple systems of a given X (cf. [11], also [7], [8] with a different notation).

A simple system $\mathcal{S}$ is called a ***filtering system*** (cf. [3], [7]– [8] and [10]– [11]) when

S5. if S_1, $S_2 \in S(x)$, then $S_1 \cap S_2 \in S(x)$, for $x \in X$.

Theorem 1. *Let $\mathcal{S}$ be a simple system in X and $\gamma_{\mathcal{S}}$ a g-topology defined by* (2)*. We have $\mathcal{S}_0^{\gamma_{\mathcal{S}}} \sqsubset \mathcal{S}$.*

If we additionally assume that $\mathcal{S}$ is filtering and that

$$\underset{S \in S(x)}{\forall} \ \underset{S_x \in S(x)}{\exists} \ \underset{y \in S_x}{\forall} S_x \in S(y), \tag{5}$$

then the two systems coincide: $\mathcal{S} = \mathcal{S}_0^{\gamma_{\mathcal{S}}}$.

Proof. Fix $x \in X$ and take $S \in S_0^{\gamma_{\mathcal{S}}}(x)$. We have $U \subset G$ for some $U \in S(x)$ and some $G \subset S$ for which $x \in G$. Thus $S \in S(x)$.

Now assume (5) and let $S \in S(x)$. Taking $U := S_x \cap S$ suffices to complete the proof by *S5.* $\square$

Assumption (5) is essential for the relation $\mathcal{S} \sqsubset \mathcal{S}_0^{\gamma_{\mathcal{S}}}$, which is clear since $\mathcal{S} \sqsubset \mathcal{S}_0^{\tau_{\mathcal{S}}}$ reads

$$\underset{S \in S(x)}{\forall} \ \underset{S_x \in S(x) \cap 2^S}{\exists} \ \underset{y \in S_x}{\forall} S_x \in S(y),$$

which is even more than (5).

We will use an operator satisfying *OP1–OP3* to construct a simple system for which (5) fails and for which we can explicitly point an $S_0 \in S(x)$ where $\mathcal{S} \sqsubset \mathcal{S}_0^{\tau_{\mathcal{S}}}$ breaks down.

Example 2. Consider an operator d satisfying conditions *OP1–OP3* and such that for some $S_0 \subset X$ and $x_0 \in X$ we have

$$S_0 \cap d\, S_0 = \{x_0\}.$$

Then, for a simple system $\mathcal{S}^d$ connected with d by means of (1), we can rewrite (2) as

$$\gamma_{\mathcal{S}^d} = \left\{ G \subset X : \underset{x \in G}{\forall} \ \underset{U \in S^d(x)}{\exists} U \subset G \right\}.$$

Consequently, by (3),

$$S_0^{\gamma_{\mathcal{S}^d}}(x) = \left\{ S \subset X : \underset{\substack{G \subset S \\ G \ni x}}{\exists} \ \underset{y \in G}{\forall} \ \underset{U \subset G}{\exists} y \in U \cap d\, U \right\}.$$

Consider S_0. Of course, $S_0 \in S^d(x_0)$. We infer that $S_0 \notin S_0^{\gamma_{\mathcal{S}^d}}(x_0)$. Indeed, let $x_0 \in G \subset S_0$. It suffices to show that

$$\exists_{\hat{x} \in G} \forall_{U \subset G} \hat{x} \notin U \cap dU.$$

We distinguish two cases.

1. If $G = \{x_0\}$, then for $U \subset G$ we have $x_0 \notin dU$. Therefore, we can put $\hat{x} := x_0$.
2. Else, there is an $x_1 \neq x_0$ in G. Let $\hat{x} := x_1$. We have $U \cap dU \subset S \cap dS = \{x_0\}$.

Thus, in both cases, $\hat{x} \notin U \cap dU$. $\diamond$

One could argue that a much simpler example could be formulated in $\mathbb{R}$ with Euclidean topology and with system defined in example 3 below. The one given above is nevertheless formulated in a great generality by which it exhibits the most crucial reason of why (5) is relevant.

Let $\mathcal{T}$ be a topology on X making X $\mathcal{T}$-dense-in-itself (i.e. there are no $\mathcal{T}$-isolated points in X). Any simple system will be refered to as a ***local system*** in $(X, \mathcal{T})$ (or simply a ***local system***, when confusion is unlikely, cf. [3], [7]– [8] and [10]– [11]), when for any $x \in X$,

S4. if $x \in U$, $U \in \mathcal{T}$ and $S \in S(x)$, then $S \cap U \in S(x)$.

Remark 1. *If $x_0 \in \mathcal{T}\operatorname{Int} U$ and $\{S(x)\}_{x \in X}$ is local, then $U \in S(x_0)$, by S4.*

Remark 2. *Requirement on X to be $\mathcal{T}$-dense-in-itself is a sine qua non condition for existence of a local system in $(X, \mathcal{T})$ for otherwise conditions S4 and S1.1 were contradictory.*

Property 3. *Generalized topology $\tau_{\mathcal{S}}$ connected with a filtering system $\mathcal{S}$ is a topology on X and X is $\tau_{\mathcal{S}}$-dense-in-itself. Moreover $\mathcal{S}$ is local in $(X, \tau_{\mathcal{S}})$.*

Proof. The first part is obvious. Note that condition $X \in \tau_{\mathcal{S}}$ is fulfilled even without filtering assumption.

X is $\tau_{\mathcal{S}}$-dense-in-itself, for $\{x\} \in \tau_{\mathcal{S}}$ would contradict *S1.1.*

Let $x \in U \subset G \in \tau_{\mathcal{S}}$ and U, $S \in S(x)$. By *S5*, $S \cap U \in S(x)$ and by *S3*, $S \cap G \in S(x)$. $\square$

Example 3. Let X be $\mathcal{T}$-dense-in-itself and d the operator assigning to A its $\mathcal{T}$-derived set. Conditions *OP1–OP3* are fulfilled and the system $\mathcal{S}^d$ is a (in fact: local) system in which $S \in S^d(x)$ iff x is in S and simultaneously a $\mathcal{T}$-limit point of S (in notation $x \in S^{der_{\mathcal{T}}}$). This system is the top of lattice of the local systems of a given space (cf. [11]) and is denoted by $\mathcal{S}_{\infty}^{\mathcal{T}}$. $\diamond$

Example 4. Let X be $\mathcal{T}$dense-in-itself and $\mathcal{F}\,\mathrm{Int}$ the operator assigning to A its $\mathcal{F}$ interior. Conditions *OP1–OP3* are fulfilled and the system $\mathcal{S}^{\mathcal{F}\,\mathrm{Int}}$ is a (in fact: local) system in which $S \in S^{\mathcal{F}\,\mathrm{Int}}(x)$ iff x is in $\mathcal{T}\mathrm{Int}\, S$. This system is the bottom of lattice of the local systems of a given space (cf. [11]). Such a system is said to be *generated by* $\mathcal{T}$ and is denoted by $\mathcal{S}_0^{\mathcal{T}}$. $\diamond$

Corollary 1. *If X is $\mathcal{T}$dense-in-itself and $\mathcal{S}$ local in $(X, \mathcal{T})$, then $\mathcal{S}_0^{\mathcal{T}} \sqsubset \mathcal{S} \sqsubset \mathcal{S}_\infty^{\mathcal{T}}$.*

Restriction of the scope to the topological spaces and local systems does not affect the special role of condition (5), as is shown in the following corollary and example.

Corollary 2. *System $\mathcal{S}_0^{\mathcal{T}}$ is local in $(X, \mathcal{T})$, filtering and fulfils* (5).

Example 5. Assume $\hat{S} \subset X$ is such that

$$\hat{S} \cap \hat{S}^{der_{\mathcal{T}}} \neq \varnothing \qquad \text{and} \qquad \hat{S} \subset \left(X \setminus \hat{S}\right)^{der_{\mathcal{T}}}.$$

There are filtering local systems in $(X, \mathcal{T})$ not generated by any topology.

Fix $\hat{z} \in \hat{S} \cap \hat{S}^{der_{\mathcal{T}}}$. Let

$$S_{\hat{z}}(x) := \begin{cases} \{S : \hat{z} \in U \cap \hat{S} \subset S, \text{ for some } U \in \mathcal{T}\}, & \text{when } x = \hat{z}, \\ S_0(x), & \text{else.} \end{cases}$$

It is clear that $\mathcal{S}_{\hat{z}} := \{S_{\hat{z}}(x) : x \in X\}$ is a filtering local system in $(X, \mathcal{T})$. In order to prove that $\mathcal{S}_{\hat{z}}$ is not generated by any topology it suffices, by Corollary 2, to find an $S_1 \in S_{\hat{z}}(\hat{z})$, so that in every $S \in S_{\hat{z}}(\hat{z}) \cap 2^{S_1}$ there is a y such that $S \notin S_{\hat{z}}(y)$. Put $S_1 := \hat{S} \cap \hat{S}^{der_{\mathcal{T}}}$ and note that $\hat{z} \in \hat{S} \cap \hat{U} \subset S$, for some $\hat{U} \in \mathcal{T}$. Since $\hat{z} \in \hat{S}^{der_{\mathcal{T}}}$, $y \in \hat{S} \cap \hat{U}$, for some $y \neq \hat{z}$. By assumption, $y \in (X \setminus \hat{S})^{der_{\mathcal{T}}}$, hence $y \notin \mathcal{F}\,\mathrm{Int}\, \hat{S}$. Therefore, $S \notin S_{\hat{z}}(y)$. $\diamond$

Remark 3. *Let $\mathcal{T}_1 \supset \mathcal{T}_2$ be two topologies on X making X a $\mathcal{T}_j$-dense-in-itself set, for $j = 1, 2$, respectively. Each local system in $(X, \mathcal{T}_1)$ is local in $(X, \mathcal{T}_2)$. However, if there are*

$$\{U_1, U_2\} \subset \mathcal{T}_2 \setminus \{\varnothing, X\} \qquad \textit{such that} \qquad U_1 \setminus U_2 \neq \varnothing,$$

then there is a $\mathcal{T}_2 \subset \mathcal{T}_1$ for which $\mathcal{S}_0^{\mathcal{T}_2}$ fails to be local in $(X, \mathcal{T}_1)$.

Proof. The first statement is trivial, for proof of the latter put $\mathcal{T}_2 := \{\varnothing, U_2, X\}$. Obviously, $\mathcal{T}_2 \subset \mathcal{T}_1$. For $x \notin U_2$ we have $S_0^{\mathcal{T}_2}(x) = \{X\}$. $\mathcal{S}_0^{\mathcal{T}_2}$ is not local in $(X, \mathcal{T}_1)$, since for $x \in U_1$ we need $U_1 \in S_0^{\mathcal{T}_2}(x)$, by Remark 1, which fails when $x \in U_1 \setminus U_2$. $\square$

Example 6. Additionally assuming $U_2 \setminus U_1 \neq \varnothing$ and putting $\mathcal{T}_0 := \{\varnothing, U_1, X\}$, with notations of the proof above, we get an interesting but simple example of two neighbourhood systems $\mathcal{S}_0^{\mathcal{T}_j}$, $j = 0, 2$, neither of which is local in the topology of the other. $\diamond$

Theorem 2. *Local systems $\mathcal{S}_0^{\tau_1}$, $\mathcal{S}_0^{\tau_2}$ in a fixed $(X, \mathcal{T})$ are comparable in the sense of relation $\sqsubset$ defined by* (4) *iff the topologies τ_1, τ_2 are comparable.*

Proof. It suffices to show that the relations $\tau_1 \subset \tau_2$ and $\mathcal{S}_0^{\tau_1} \sqsubset \mathcal{S}_0^{\tau_2}$ are equivalent.

First assume that $\mathcal{S}_0^{\tau_1} \sqsubset \mathcal{S}_0^{\tau_2}$ and fix a $U \in \tau_1$. We have $U \in S_0^{\tau_1}(x) \subset S_0^{\tau_2}(x)$, for $x \in U$. Hence, for $x \in U$ there is a $U_x \in \tau_2$ such that $x \in U_x \subset U$. Therefore $U = \bigcup_{x \in U} U_x \in \tau_2$, and the assertion follows.

The converse implication is trivial. □

Corollary 3. *The relations $\mathcal{S}_0^{\tau_1} = \mathcal{S}_0^{\tau_2}$ and $\tau_1 = \tau_2$ are equivalent. Moreover $\mathcal{S}_0^{\tau_\mathcal{S}} = \mathcal{S}$ and $\tau_{\mathcal{S}_0^\tau} = \tau$.*

Note 1. A condition 'slightly stronger' than (5) originally appeared in [9], Theorem 4.2, pp. 31–32. The question, whether one can remove quotation marks here or not seems to an open problem.

Note 2. In [2] author goes even further in generalizing neighbourhood systems, and requires only one condition, *S2* here. Lemmas 1.2 and 1.3 of [2] establish a correspondence between generalized neighbourhood systems and g-topologies. Thus, the first part of our Theorem 1 is a straightforward consequence of these.

Note 3. Our notation is a little inconsistent with [7], where the notion of a simple system coincides with that of a local system from [8]. Nonetheless the author considers himself excused, as B. Thomson, the inventor of local systems, is equally inconsistent with his own notation.

Note 4. A parallel theory can be developed with a ***vicinity system*** as a starting point. From such a system we would require the following conditions:

VS1.1. $\{x\} \notin S(x)$, *VS1.2.* $S(x) \neq \varnothing$,

VS2. if $S \in S(x)$, then $S \setminus \{x\} \in S(x)$,

VS3. if $S_1 \in S(x)$ and $S_1 \subset S_2$, then $S_2 \in S(x)$.

There is a one to one correspondence between the vicinity systems (here: $\mathcal{S}_v = \{S_v(x)\}_{x \in X}$) and the simple systems (here: $\mathcal{S}_s = \{S_s(x)\}_{x \in X}$), established by the following formulae:

$$S_v(x) := \{S : S \cup \{x\} \in S(x)\}, \qquad \text{where } \{S(x)\}_{x \in X} \text{ forms a simple system,}$$
$$S_s(x) := \{S \cup \{x\} : S \in S(x)\}, \qquad \text{where } \{S(x)\}_{x \in X} \text{ forms a vicinity system.}$$

Of course, we have $(\mathcal{S}_v)_s = \mathcal{S}$ and $(\mathcal{S}_s)_v = \mathcal{S}$.

Note 5. The question about a relation in the spirit of Property 1 but restricted to local systems is partially answered in [10]. One can get a local system through formula (1) providing the operator $d : 2^X \to 2^X$ is such that:

$$\mathcal{T}\,\mathrm{Int}\,A \subset d\,A \subset A^{der_{\mathcal{T}}}, \qquad d\,(A \cap B) = d\,A \cap d\,B,$$

for A, $B \subset X$.

Note 6. For the operator $F_{\mathcal{S}} : 2^X \to 2^X$ considered in [10] and defined as

$$F_{\mathcal{S}}(A) := \{x \in X : A \in S(x)\}, \qquad \text{for} \qquad A \subset X,$$

we get one more formula for interior in the topology related to $\mathcal{S}$ (cf. properties 2 and 3 here), namely:

$$\tau_{\mathcal{S}}\text{-}\,\mathrm{Int}\,A = \bigcup \{G : G \subset A \cap F_{\mathcal{S}}G\}\,.$$

Note 7. By the very same method as in the proof of Remark 3 we can prove that neither $\mathcal{S}_{\infty}^{\mathcal{T}_{\mathrm{I}}}$ is local in $(X, \mathcal{T}_{\mathrm{I}})$.

References

[1] Á. Császár, *Generalized open sets*, Acta Math. Hungar. **75**, no. 1–2 (1997), 65–87.

[2] —, *Generalized topology, generalized continuity*, Acta Math. Hungar. **96**, no. 4 (2002), 351–357.

[3] V. Ene, *Real Functions — Current Topics*, Lecture Notes in Mathematics, Springer, 1995.

[4] T. R. Hamlett, *A correction to the paper "Semi-open sets and semi-continuity in topological spaces" by Norman Levine*, Proceedings of the AMS **49**, no. 2 (1975), 458–460.

[5] N. Levine, *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly **70** (1963), 36–41.

[6] O. Njåstad, *On some classes of nearly open sets*, Pacific Journal of Math. **15**, no. 3 (1965), 961–970.

[7] B. Thomson, *Derivation bases on the real line II*, Real Anal. Exchange **8** (1982–83), 278–442.

[8] —, *Real Functions*, Lecture Notes in Mathematics, Springer-Verlag, 1980.

[9] S. Willard, *General Topology*, Addison-Wesley, 1970.

[10] R. Zduńczyk, *Local systems and some classes of operators*, in: Real Functions, Density Topologies and Related Topics, Wydawnictwo Uniwersytetu Łódzliego, Łódź 2011, 183–189.

[11] —, *Generalized discontinuity of real-valued functions*, Tatra Mt. Math. Publ. **55** (2013), 1–16.

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PROSTE SYSTEMY I UOGÓLNIONE TOPOLOGIE

S t r e s z c z e n i e

Wprowadzamy pojęcie prostego systemu jako uogólnienie systemu lokalnego Thomsona. Za jego pomocą konstruujemy uogólnioną topologię. W przypadku systemu filtrującego konstrukcja prowadzi do topologii w klasycznym sensie, w której dany zbiór jest w sobie gęsty. Badamy zależności między systemami lokalnymi i powiązanymi topologiami.

Słowa kluczowe: lokalny system, uogólnione topologie

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REMARKS ON ROUND METRIC SPACES

Summary

A metric d is called round if any closed ball $B_d[x,r]$ is a closure of open ball $B_d(x,r)$. We present theorems concerning space of bounded functions and Hausdorff metric. We also introduce the notion of strongly not round metric and show that for some metric spaces (X,d) there exists such metric equivalent to d.

Keywords and phrases: round metric, Hausdorff metric

Let (X,d) be a metric space with at least two points. Let $\overline{A}$ denotes a closure of set A. For $x \in X$ and $r > 0$, let

$$B_d(x,r) = \{y \in X : d(x,y) < r\} \quad \text{and} \quad B_d[x,r] = \{y \in X : d(x,y) \leq r\}.$$

Definition 1. *A metric d is called round if for each $x \in X$ and $r > 0$ we have*

$$\overline{B_d(x,r)} = B_d[x,r].$$

We say that a metric space (X,d) is round if metric d is round. Any real or complex normed vector space with metric induced by norm is round. Subspace $X = (1,2) \cup (3,4)$ of $\mathbb{R}$ is round with natural metric inherited from $\mathbb{R}$. Any discrete metric space is not round.

For any metric space (X,d) there exists a metric d_1 equivalent to d and not round. If $X = A \cup K$ is a metrizable space, where A and K are nonempty, disjoint, closed sets, and K is compact, then no metric for X can be round. For the proofs we refer the reader to [1].

Let $F(X,(Y,d))$ be a space of bounded functions from a fixed set X to a metric space (Y,d). Define metric $\widehat{d}$ by

$$\widehat{d}(f,g) = \sup_{x \in X}\{d(f(x), g(x))\},$$

for $f, g \in F(X, (Y, d))$. It is easy to check that if (Y, d) is round then space $F(X, (Y, d))$ may not by round. It is enough to take $Y = (0,1) \cup (2,3)$, $X = (0,1)$ and consider functions $f_1(x) = x$ and $f_2(x) = x + 2$. It is easy to see that

$$f_2 \notin \overline{B_{\widehat{d}}(f_1, \widehat{d}(f_1, f_2))}.$$

In this work we give a condition which implies roundness of $F(X, (Y, d))$. We show that roundness of metric d is equal to roundness of Hausdorff metric induced by d. We also introduce a notion of strongly not round metric and show that for some metric spaces there exists equivalent metric which is strongly not round.

We begin with a few technical lemmas. It is easy to check that.

Lemma 1. *Metric d is round if and only if for all $x, y \in X$ with $x \neq y$ we have*

$$y \in \overline{B_d(x, d(x, y))}.$$

Lemma 2. [1] *If $\alpha \in (0,1)$ and (X, d) is a metric space such that for all $x, y \in X$ there exists $z \in X$ such that*

$$d(z, x) = (1-\alpha)d(x, y) \quad \text{and} \quad d(z, y) = \alpha d(x, y), \tag{1}$$

then d is round.

Lemma 3. *Let (X, d) be a metric space satisfying (1). For all R, S, r such that $R > 0$ and $0 < S < r$ there exists $n_0 \in \mathbb{N}$ such that for any $a, b \in X$ with*

$$S \leq d(a, b) \leq r,$$

there exists $b' \in X$ such that

$$d(b, b') < R \qquad \text{and} \qquad d(a, b') < r - \frac{1}{n_0}.$$

Proof. By (1) we can create a sequence $(z_k)_{k \in \mathbb{N}}$ tending to b such that for each $n \in \mathbb{N}$ we have $d(z_n, b) = \alpha d(a, b)$ Let a pair of natural numbers k_0, n_0 satisfies

$$\frac{1}{Sn_0} < \alpha^{k_0} < \frac{R}{r}.$$

Let $b' = z_{k_0}$. Therefore

$$d(b', b) = \alpha^{k_0} d(a, b),$$

thus

$$d(b', b) < \frac{d(a,b)R}{r} \leq \frac{rR}{r} = R \tag{2}$$

and

$$\frac{1}{n_0} = \frac{S}{Sn_0} \leq \frac{d(a,b)}{Sn_0} < d(b', b),$$

which gives

$$d(b', a) \leq (1 - \alpha^{k_0}) d(a, b) = d(a, b) - d(b', b) < r - d(b', b) < r - \frac{1}{n_0}. \tag{3}$$

By (2) i (3) the proof is complete. □

Theorem 1. *Suppose that* (Y, d) *satisfies* (1). *Metric space* $F(X, (Y, d))$ *with a metric* $\widehat{d}$ *is round if and only if metric* d *is round.*

Proof. Suppose that d is round. Choose $f, g \in F(X, (Y, d))$ with $f \neq g$. Let $r = \widehat{d}(f, g)$. We will prove that

$$f \in \overline{B_{\widehat{d}}(g, \widehat{d}(f, g))}.$$

Let $\varepsilon > 0$. We look for a function $z \in F(X, (Y, d))$ such that $z \in B_{\widehat{d}}(f, \varepsilon)$ and $z \in B_{\widehat{d}}(g, \widehat{d}(f, g))$.

To define a function z, choose numbers S and R satisfying $0 < R < \varepsilon$ and $0 < S < r$. By lemma, there exists n_0 such that for any $a, b \in Y$ with

$$S \leq d(a, b) \leq r,$$

there is $b' \in Y$ such that

$$d(b, b') < R \qquad \text{and} \qquad d(a, b') < r - \frac{1}{n_0}.$$

Let $x \in X$. If $d(f(x), g(x)) < S$, then we put $z(x) = f(x)$. Otherwise, we find y' satisfying

$$d(f(x), y') < R \qquad \text{and} \qquad d(g(x), y') < r - \frac{1}{n_0}$$

and put $z(x) = y'$. Clearly, z is bounded. Moreover, for each $x \in X$ we have $d(f(x), z(x)) < R$. Thus

$$\widehat{d}(z, f) = \sup_{x \in X} \{d(f(x), z(x))\} \leq R < \varepsilon. \tag{4}$$

Additionally, for any $x \in X$,

$$d(z(x), g(x)) < \max\{S, r - \frac{1}{n_0}\}.$$

Therefore

$$\widehat{d}(z, g) = \sup_{x \in X} \{d(g(x), z(x))\} \leq \max\{S, r - \frac{1}{n_0}\} < r. \tag{5}$$

By (4) and (5) we have

$$z \in B_{\widehat{d}}(f, \varepsilon)$$

and

$$z \in B_{\widehat{d}}(g, \widehat{d}(f, g)).$$

Let us now suppose that $\widehat{d}$ is round. We choose $a, b \in X$ such that $a \neq b$. Let $f(x) = a$ and $g(x) = b$ for $x \in X$. Functions f, g are bounded and so there exists a sequence $(z_n)_{n\in\mathbb{N}}$ of bounded functions tending to f and such that $z_n \in B_{\widehat{d}}(g, \widehat{d}(f,g))$. Let us choose $x_0 \in X$ and consider the $(z_n(x_0))_{n\in\mathbb{N}}$. Since

$$d(z_n(x_0), a) \leq \widehat{d}(z_n, f) \text{ for each } n \in \mathbb{N},$$

$(z_n(x_0))_{n\in\mathbb{N}}$ tends to a. Moreover, for all $n \in \mathbb{N}$ we have

$$d(b, z_n(x_0)) \leq \widehat{d}(g, z_n) < \widehat{d}(f,g) = d(a,b).$$

Thus $(z_n(x_0))_{n\in\mathbb{N}}$ lies in $B_d(a, d(a,b))$. Consequently,

$$b \in \overline{B_d(a, d(a,b))}.$$

Therefore d is round. □

Let (X, d) be a metric space. Let h be the Hausdorff metric defined in the space of all compact subsets of X by

$$h(A,B) = \max\{\sup_{x\in A}\{d(x,B)\}, \sup_{x\in B}\{d(x,A)\}\},$$

for $A, B \in C(X)$.

Theorem 2. *The Hausdorff metric h is round if and only if the metric d is round.*

Proof. Suppose that d is round. Let $F, G \in C(X)$ and $\varepsilon > 0$. Firstly, we will prove that

$$\bigcup_{x\in F} \overline{B_d(x,\varepsilon)} = \overline{\bigcup_{x\in F} B_d(x,\varepsilon)}. \tag{6}$$

The inclusion

$$\bigcup_{x\in F} \overline{B_d(x,\varepsilon)} \subset \overline{\bigcup_{x\in F} B_d(x,\varepsilon)}$$

is obvious. If

$$y_0 \in \overline{\bigcup_{x\in F} B_d(x,\varepsilon)}$$

then y_0 is a limit of a sequence from

$$\bigcup_{x\in F} B_d(x,\varepsilon).$$

For each $n \in \mathbb{N}$ we choose $x_n \in X$ such that $y_n \in B_d(x_n, \varepsilon)$. By the compactness of F there exists $x_0 \in F$ such that some subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ is convergent to x_0. Hence for each $k \in \mathbb{N}$ we have

$$d(x_{n_k}, y_{n_k}) < \varepsilon \quad \text{and so} \quad d(x_0, y_0) \leq \varepsilon.$$

Since d is round it follows that

$$y_0 \in B_d[x_0, \varepsilon] = \overline{B_d(x_0,\varepsilon)} \subset \bigcup_{x\in F} \overline{B_d(x,\varepsilon)}.$$

Our next claim is that

$$(7)\qquad (G \in \overline{B_h(F,\varepsilon)}) \iff (G \subset \bigcup_{x\in F} \overline{B_d(x,\varepsilon)} \wedge F \subset \bigcup_{x\in G} \overline{B_d(x,\varepsilon)}).$$

Indeed we have

$$\begin{aligned}
(G \not\subset \bigcup_{x\in F} \overline{B_d(x,\varepsilon)}) \quad\wedge\quad (F \not\subset \bigcup_{x\in G} \overline{B_d(x,\varepsilon)}) &\quad \Leftrightarrow\\
(\exists_{\substack{x_1\in G\\ r_1>0}} B_d(x_1,r_1) \subset X \setminus \bigcup_{x\in F} B_d(x,\varepsilon)) \vee (\exists_{\substack{x_2\in G\\ r_2>0}} B_d(x_2,r_2) \subset X \setminus \bigcup_{x\in G} B_d(x,\varepsilon)) &\quad \Leftrightarrow\\
(\exists_{\substack{x_1\in G\\ r_1>0}} r_1 + \varepsilon \le d(x_1,F) \le h(G,F)) \vee (\exists_{\substack{x_2\in G\\ r_2>0}} r_2 + \varepsilon \le d(x_2,G) \le h(G,F)) &\quad \Leftrightarrow\\
G \notin \overline{B_h(F,\varepsilon)}. &
\end{aligned}$$

In the same manner we can check that

$$(8)\qquad (G \in B_h[F,\varepsilon]) \iff (G \subset \bigcup_{x\in F} B_d[x,\varepsilon] \wedge F \subset \bigcup_{x\in G} B_d[x,\varepsilon]).$$

From (6), (7), (8) and roundness of d we conclude that

$$B_h[F,\varepsilon] = \overline{B_h(F,\varepsilon)}.$$

Suppose now that h is round and fix two distinct points $x, y \in X$ Since $\{x\}$ and $\{y\}$ are compact, there exists a sequence $\{A_n\}_{n\in\mathbb{N}}$ convergent to $\{y\}$ such that $A_n \in B_h(\{x\}, h(\{x\},\{y\}))$. Let $a_n \in A_n$ for each $n \in \mathbb{N}$. We have

$$d(a_n, y) \le \sup_{a\in A_n} \{d(a,\{y\})\} \le \max\{\sup_{a\in A_n} \{d(a,\{y\})\}, d(y, A_n)\} = h(\{y\}, A_n).$$

Thus

$$\lim_{n\to\infty} d(a_n, y) = 0.$$

Moreover for all $n \in \mathbb{N}$ we have

$$\begin{aligned}
d(a_n, x) \le \sup_{a\in A_n} \{d(a,\{x\})\} \le \max\{\sup_{a\in A_n} \{d(a,\{x\})\}, d(x, A_n)\} = \\
h(\{x\}, A_n) < h(\{x\},\{y\}).
\end{aligned}$$

Therefore $a_n \in B_d(x, d(x,y))$ for any $n \in \mathbb{N}$. Consequently,

$$y \in \overline{B_d(x, d(x,y))}$$

and d is round. □

The letter proof does not work for the Hausdorff metric defined for all nonempty and bounded subsets of X. Indeed let $X = (0,1) \cup (2,3)$. Then for each $x \notin (0,1)$ we have $d(x,(0,1)) > 1$. Thus for all sets $B \subset X$ such that $h(B,(0,1)) < 1$, must be $B \subset (0,1)$. Therefore if we take sequence $(A_n)_{n\in\mathbb{N}}$ convergent to $(0,1)$, then for $n \in \mathbb{N}$ grater then some $n_0 \in \mathbb{N}$ we have $A_n \subset (0,1)$. We thus get

$$\sup_{a\in A_n} \{d(a,(2,3))\} \ge \sup_{a\in(0,1)} \{d(a,(2,3))\}$$

and

$$\sup_{x\in(2,3)}\{d(x,A_n)\} \geq \sup_{x\in(2,3)}\{d(x,(0,1))\}.$$

Therefore

$$h(A_n,(2,3)) \geq h((0,1),(2,3)),$$

hence

$$A_n \notin K_{\widehat{d}}((0,1),h((0,1),(2,3)))$$

and consequently

$$(2,3) \notin \overline{B_h((0,1),h((0,1),(2,3)))}.$$

Now we will give a definition of strongly not round metric space.

Definition 2. *We say that (X,d) is strongly not round if for any $x\in X$ and $R>0$ there exists $0<r<R$ such that*

$$\overline{B_d(x,r)} \neq B_d[x,r].$$

Theorem 3. *Let (X,d) satisfies* (1)*. There exists metric e equivalent to d and strongly not round.*

Proof. Consider a function

$$f(x) := \begin{cases} C(x), & \text{if } \ x\in[0,1],\\ 1, & \text{if } \ x\in(1,\infty).\end{cases}$$

Where $C(x)$ is a Cantor function. Define

$$e(x,y) = f(d(x,y)) \quad \text{for} \quad x,y\in X.$$

The function e is a metric. We shall prove the triangle inequality. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence defined by

$$f_1(x) := \begin{cases} x, & \text{if } \ x\in[0,1],\\ 1, & \text{if } \ x\in(1,\infty).\end{cases} \qquad f_{n+1}(x) := \begin{cases} \frac{1}{2}f_n(3x), & \text{if } \ x\in[0,\frac{1}{3}],\\ \frac{1}{2}, & \text{if } \ x\in(\frac{1}{3},\frac{2}{3}],\\ \frac{1}{2}+\frac{1}{2}f_n(3x-2), & \text{if } \ x\in(\frac{2}{3},1],\\ 1, & \text{if } \ x\in(1,\infty).\end{cases}$$

One can check that $\{f_n\}_{n\in\mathbb{N}}$ converges to f. Observe, that for any n, the function f_n on subintervals of $[0,1]$ is constant or linear (with the same slope on each). Hence, if $a,b\in X$, $a\leq b$, f_n then

$$f_n(b-a) \geq f_n(b) - f_n(a). \tag{9}$$

Therefore for all $x,y\in[0,\infty)$ and $n\in\mathbb{N}$

$$f_n(x+y) \leq f_n(x) + f_n(y).$$

Applying (9) to $b = x + y$ and $a = x$, we obtain

$$f(x + y) \leq f(x) + f(y).$$

Consequently, for any $x, y, z \in X$,

$$e(x, y) = f(d(x, y)) \leq f(d(x, z)+d(z, y)) \leq f(d(x, z))+f(d(z, y)) = e(x, z)+e(z, y).$$

Equivalence of metrics is obvious. In order to prove that e is strongly not round we fix $x \in X$ and $\varepsilon > 0$. Let

$$A = \{d(x, y) : y \in X\}.$$

From (1) it follows that A is dense in $[0, s]$ where

$$s = \sup\{|a - b| : a, b \in A\}.$$

Then there exists $y \in X$ such that

$$f(d(x, y)) < \varepsilon \quad \text{and} \quad d(x, y) \in (\frac{1}{3^n}, \frac{2}{3^n})$$

for some $n \in \mathbb{N}$. Therefore

$$B_e(x, f(d(x, y))) = B_e(x, e(x, y)) \subset B_e(x, \varepsilon).$$

Moreover,

$$B_d(x, \frac{1}{3^n}) = B_e(x, e(x, y)).$$

Since e and d are equivalent,

$$\overline{B_d(x, \frac{1}{3^n})} = \overline{B_e(x, e(x, y))}.$$

Clearly

$$y \notin \overline{B_d(x, \frac{1}{3^n})} = \overline{B_e(x, e(x, y))}.$$

Therefore

$$\overline{B_e(x, e(x, y))} \neq B_e[x, e(x, y)].$$

$\square$

References

[1] M. B. Nathanson, *Round metric spaces*, The American Mathematical Monthly **8**, no. 7 (1975), 738–741.

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UWAGI O PRZESTRZENIACH METRYCZNYCH ZAOKRĄGLONYCH

S t r e s z c z e n i e

Metryka d jest zaokrąglona wtedy i tylko wtedy, gdy kula domknięta $B_d[x, r]$ jest domknięciem kuli otwartej $B_d(x, r)$. W pracy prezentowane są twierdzenia dotyczące metryki Hausdorffa oraz przestrzeni funkcji ograniczonych. Przedstawione jest pojęcie metryki silnie niezaokrąglonej, a także dowód twierdzenia mówiącego, że dla pewnych przestrzeni metrycznych (X, d) istnieje metryka silnie niezaokrąglona oraz równoważna d.

Słowa kluczowe: metryka zaokrąglona, metryka Hausdorffa

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pp. 65–74

In memory of
Professor Claude Surry

Stanisław Bednarek

UNIVERSAL TORNADO TUBE

Summary

A few of new experiments with using of the improved tornado tube were described here. This tube including two plastic, transparent bottles, whose outlets are connected with a ball valve, applied in water supply networks. After being filled with low-viscosity fluid, this tube allow to demonstrate the process of generation, dampening and reanimation (self-excitation) of a tornado. When the fluid is replaced, the tube enables investigation of foam structure and of motion in viscous medium. If the instrument is filled with non-Newtonian fluid, the Fano effect may be investigated.

Keywords and phrases: vortex, bottle, fluid, viscosity, tornado, foam, effect Fano

1. Introduction

An experiment entitled “tornado in a bottle” is widely known. This commonness is indicated by its numerous descriptions on the Internet [1]. In this experiment, a instrument that is built from two transparent plastic bottles is applied (Fig. 1). One of the bottles is filled with water. Ends of bottles are directed towards each other and joined with a pipe or a connector, which may be constructed on one’s own or bought in a shop with education toys [2]. In order to conduct the experiment, the bottle filled with water needs to be directed upwards, and the instrument should be quickly turned (“rotated”) several times around the vertical axis. Then, an effective vortex, similar to the tornado, will be generated in the upper bottle. This is the reason why the described instrument is called a “tornado tube”. The aim of this article is to demonstrate an improved version of this tube, which allows to conduct new experiments.

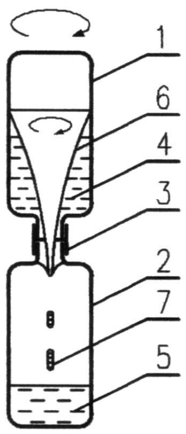


Fig. 1: Construction of the well-known tornado tube; 1, 2 – transparent plastic bottles – upper and lower one, 3 – connector, 4, 5 – water in upper and lower bottle, 6 – surface of the vortex, 7 – water flowing downwards.

2. Construction of the universal tornado tube

Improvement to the tornado tube comprises of application of a ball valve (used in water supply networks) to join the two bottles. This valve allows to regulate the fluid's speed of flow to the lower bottle or to close this flow completely. External view of the improved tornado tube is presented on Fig. 2 and 3. The instrument may be constructed from transparent plastic bottles, with capacity of 0.5–2.5 l. The bottles should present even side surface, without significant narrowings or depths. The bottles may be joined together with ball valves applied in water supply networks and adjusted to tubes with diameter of 0.5–2 inches. The most advantageous method of connection is to join the bottles with the valve in a way that enables them to become twisted off. This allows to replace the fluid in the tube and bottles easy. For that purpose, bottle's caps are connected with the valve permanently, and bottles' outlet may be twisted into these caps. This method of connecting the caps with the ball valve is demonstrated on Fig. 4.

Most of the bottles with beverages has a standard outlet with an internal diameter 25 mm and external diameter of the cap equal to 32 mm. Then, a valve of higher diameter may be applied, e.g. 1.75 inches, and the caps may be pushed into threaded valve seats, that are used for connection with pipes of water supply network (Fig. 4a). Before pushing it into, holes of diameter equal with the diameter of the hole of the valve's ball and should be cut out, and the seats should be covered with epoxy glue. In case of some caps it may be necessary to slightly decrease their internal diameter or to increase the diameter of the valve's seat with a file or a scrapper. Another method of joining the caps with the valve together comprises of cutting out holes in them, which will allow to place them on the valve tips (Fig. 4b). External surfaces of the tips need to be filed beforehand so they have a cylindrical (not hexagonal shape). What is more, the caps also need to be glued to the valve with epoxy glue. Additionally, the connection between the caps may be strengthened through rings cut out from aluminum or plastic sheet and glued inside the caps. The second method of connection is appropriate if bottles with larger dimension of the end – e.g. from Nestea drinks (internal diameter of the outlet is 38 mm and the external diameter of the cap is 42 mm) – are applied together with a valve that presents lower diameter. Furthermore, the bottles may be joined permanently to the valve through pushing their outlets directly into valve seats covered with epoxy glues or sealing and gluing paste, e.g. Poxilina. A minus of this method is the fact that it is impossible to replace the fluid, therefore one bottle should be filled with proper liquid beforehand.

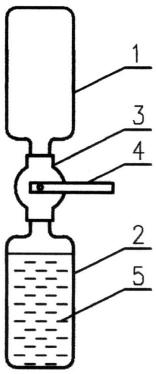


Fig. 2: Construction of the universal tornado tube; 1, 2 – transparent plastic bottles – upper and lower, 3 – body of the ball valve, 4 – lever of the ball valve.

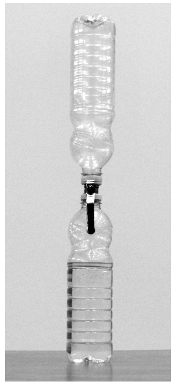


Fig. 3: External view of one of the constructed universal tornado tubes.

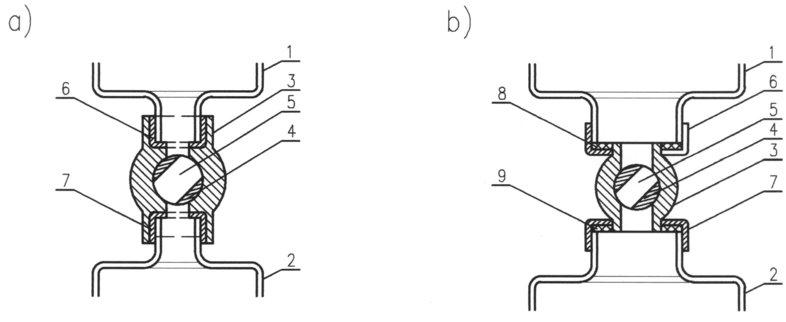


Fig. 4: Connection between the ball valve with bottles: a) with caps pushed into, b) with caps placed on, 1, 2 – transparent plastic bottles – upper and lower, 3 – body of the ball valve, 4 – ball, 5 – a hole in the ball, 6, 7 – caps of the lower and upper bottles, 8, 9 – rings.

3. Damping and reanimation tornado

Generation of the tornado in the universal tube takes place in a similar way as the one described in the introduction. It is worth conducting the first experiments with a bottle filled with water. Before putting the tube into rotary motion, the ball valve needs to be completely open (Fig. 5a, 6a). When a stable vortex is generated, the valve needs to be closed. Then, lower part of the vortex will decay and depth, which is reached by the vortex, will be decreased. Therefore, vortex walls become less inclined (Fig. 5b, 6b).

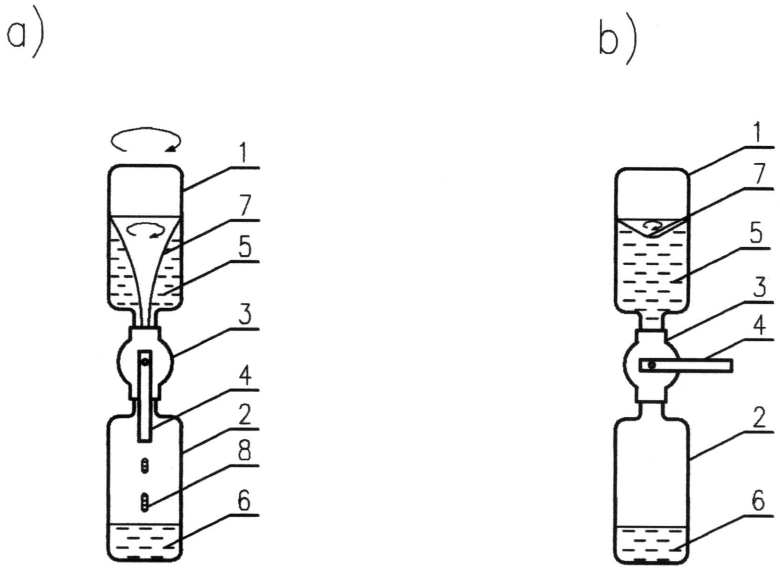
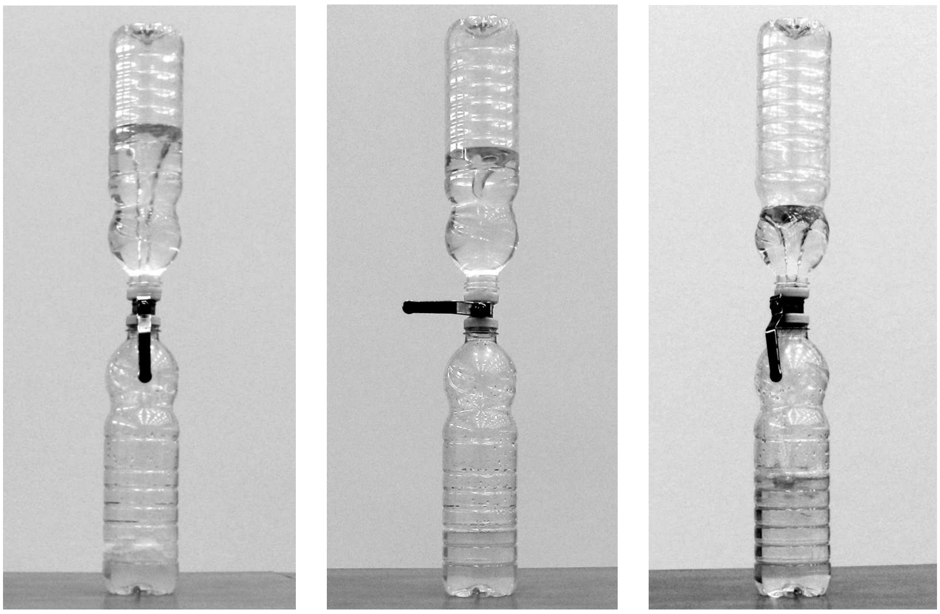


Fig. 5: Formating and dampening of the tornado in the universal tube; 5, 6 – fluid of low viscosity in the upper and lower bottle, 7 – surface of the vortex, 8 – fluid flowing downwards, digits 1–4 determine the same elements as on Fig. 2.

Instability of the lower end of vortex (which moves along a wave line) may be sometimes observed. Upper surface of the fluid may even become almost flat. If after not a long time the valve was opened, the vortex would appear again without a need to turn the bottle (Fig. 6c). The described actions may be repeated several times and the tornado may be reanimated with a decreasing amount of water, until the upper bottle becomes completely empty. In the next phase of experiment, water needs to be replaces with fluid of higher viscosity, e.g. oil or glycerin, and then with fluid of lower viscosity, e.g. ethyl alcohol. After repeating the experiments with fluids of higher viscosity, it will turn out that it is a way harder to generated a vortex in such a case. The tube needs to be put in faster rotational motion. When the valve becomes closed, the generated vortex disappears quicker and it is tougher to reanimate it. Reverse regularities are present in case of fluids with lower viscosity. These experiments prove that in order to provide a stable vortex, fluid needs to flow in a vertical direction. Owing to that fact, a part of potential energy of the fluid transforms into its kinetic energy. This compensates dissipation of initial kinetic energy of the fluid, caused by viscosity forces.



a) b) c)

Fig. 6: Subsequent phases of tornado dampening and reanimation; a) initial tornado, b) tornado damped trough closing the valve, c) reanimated tornado.

4. Shape of the vortex's free surface

If we accept a frame of reference connected with the fluid in motion, then any element with mass Δm is influenced by the following forces: centrifugal force $\mathbf{F}_r$, weights W and force of viscous resistance (Fig. 7). For the fluid of insignificant viscosity, the last parameter may be omitted. The free surface of the vortex takes on a form, for which the resultant of forces $\mathbf{F}_r$ and W, i.e. the $\mathbf{F}$ force is perpendicular to this surface in every point [3]. The rotation velocity of fluid increases together with approaching the lower end of the vortex. This is caused by lowering the radius of circle, which is followed by the fluid, and transforming a part of a potential energy of weight into kinetic energy at the time when the fluid flows into the lower part of the vortex. Precise determination of $f(r)$ function, which describes the shape of the vortex free surface with consideration of fluid viscosity, would demand application of Naiver-Strokes equation which describes motion of viscous fluid [4]. This leads to a very complicated differential equation, which goes beyond the scope of this article [5]. Similarly complicated differential equation are required in case using of Bernoulli's equation.

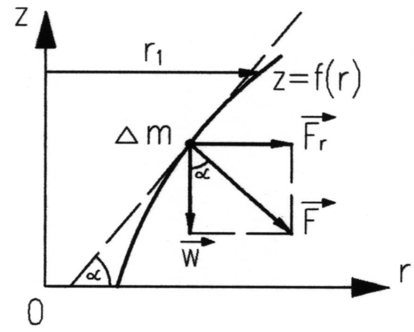


Fig. 7: Forces acting at surface of the liquid in the tornado tube; F_r – centrifugal force, W – weight, F – resultant force, Δm – weight of fluid element.

The situation becomes very simplified when it is assumed that transformation of the fluid's potential energy compensates the dissipation of its kinetic energy caused by viscosity forces [6]. Then, speed of the fluid element on the vortex surface may be calculated according to the law of angular momentum conservation.

$$\Delta m v_1 r_1 = \Delta m v r. \tag{1}$$

In equation (1) v_1 and r_1 mean respectively initial speed and radius of the circle, which is followed by the element of weight Δm. Centrifugal and weight forces are expressed in the following formulas:

$$F_r = \frac{\Delta m v^2}{r}, \tag{2}$$

$$W = \Delta m g. \tag{3}$$

What is more, the following equation results from Fig. 7.

$$\tan\alpha = \frac{F_r}{W} = \frac{dz}{dr}. \tag{4}$$

In result substitution of equation (1) to (2), and next (2) and (3) to (4), the following differential equation is obtained

$$\frac{dz}{dr} = \frac{(v_1 r_1)^2}{g}\left(\frac{1}{r^3}\right). \tag{5}$$

Equation (5) is easily solved through direct integration, and the following formula is obtained

$$z(r) = -\frac{(v_1 r_1)^2}{2g}\left(\frac{1}{r^2}\right) + C. \tag{6}$$

Letter C in equation (6) stands for a constant of integration, which depends on the initial height of fluid column in the upper bottle. It results from equation (6) that free surface of vortex takes a form of rotary hyperboloid. This result needs to be verified in case of particular fluid. It may be done through taking a photo of the vortex's surface, which was generated during the experiment, measuring value of coordinates of points on axial section and comparing them with coordinates calculated from the equation (6).

5. Motion in viscous fluid and foam structure

Universal tornado tube enables generation of single air bubbles that move in viscous fluid. For that purpose, the tube needs to be put vertically, so one bottle filled with the investigated fluid of high viscosity, e.g. oil or glycerin, is located at the top. The ball valve should be closed then. Through partially opening the valve for a short moment, a slight air bubble is introduced into the upper bottle (Fig. 8).

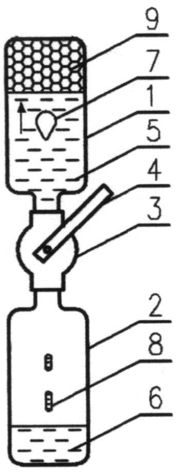


Fig. 8: Application of the universal tornado tube for investigation of foam structure and of motion in a viscous fluid; 5, 6 – shampoo in upper and lower bottle, 7 – air bubble, 8 – shampoo flowing down, 9 – foam, digits 1–4 determine the same elements as on Fig. 2.

This bubble is influenced by three forces: weight of included air, lift force and force of viscous resistance, which is expressed with Stokes equation and which is directly proportional to the velocity. With proper velocity, these forces become compensated, and the bubble moves with a uniform motion in upward direction. Motion of those bubbles may also be investigated quantitatively, which allows to determine the fluid viscosity coefficient. In order to obtain the highest accuracy, possibly high bottles need to be applied and motion of the bubble should be examined at some distance from the valve, when its velocity is constant. The upper bottle may be filled with highly viscous fluid, capable of foam generation, e.g. with shampoo or dishwash. When the valve is partially open, the air bubbles go through that fluid and generate foam on its top surface. Gradually, the foam fills the whole upper bottle (Fig. 9).

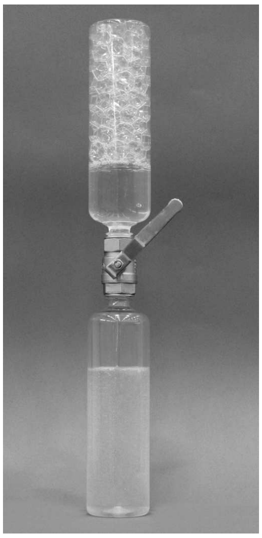


Fig. 9: Foam formed in the universal tornado tube after the shampoo flow.

This foam presents highly regular cells and lasts for several hours, which allows to investigate its structure. Then, walls that separate the foam cells become thinner as the fluid flows downwards. The cells burst and connect with each other. This process starts from the top and the foam slowly disappears. The initial size of the foam cells and speed of their formation depends on the part, to which the valve is open.

6. Investigation of the Fano effect

Fluids, whose coefficient of viscosity does not depend on the velocity are called Newtonian fluids [7]. However, there are some fluids that do not meet the mentioned condition. Such fluids are called non-Newtonian or rheology fluids. Examples of rheology fluids encompass: honey, suspension of potato starch in water, ketchup and epoxy resin. They present interesting phenomenas that are not observed in case of Newtonian fluids. One of such phenomenas is the Fano effect [8]. In order to observe this effect, one bottle of the tube needs to be filled with synthetic honey. Natural honey may also be applied, but it is much more expensive, and after some time it will become solidified as a result of precipitation of sugar (so called sugaring). The universal tornado tube needs to be placed vertically, so the bottle with honey is on the top, then the valve should be slightly opened. In such a situation, a slow motion of a thin trickle of honey towards the lower bottle is observed. This trickle does not flow perfectly vertically, but firstly it rotates around the cone surface and accumulates on the honey surface in the lower bottle, and then it melts away. Simultaneously, motion of air bubbles present in honey located in the upper bottle can be observed. When the bubble reaches the honey surface in this bottle, outlet of the trickle in the lower bottle becomes thicker and faster for a short time. This is caused by an increase of air pressure above the honey surface in the upper bottle. A cause of the Fano effect is the fact that there are perpendicular tensions appearing in the direction of motion.

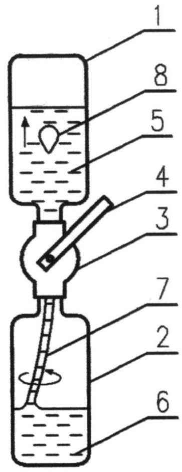


Fig. 10: Application of the universal tornado tube to investigation the Fano effect; 5, 6 – non-Newtonian fluid in the upper and lower bottle, 7 – rotating column of the fluid, 8 – air bubble, digits 1–4 determine the same elements as on Fig. 2.

Fig. 11: Fano effect observed in the universal tornado tube.

7. Conclusions

Important advantages of the described instrument comprise of: low cost of necessary materials, easiness in construction and multi-functionality that allows to use it for presentation and quantitative investigation of various phenomena. Both presentations and the investigations can be easily repeated. It is enough to turn the instrument for 180°. The fact that the fluid is enclosed in a system of bottles secures the researchers against getting wet and dirty, which is especially important in case of using fluid with high viscosity, e.g. oil and shampoo. Therefore, the described instrument is appropriate for both demonstration of physical phenomena and laboratory research during classes with students. Theses demonstrations may also have interactive character, and they may be carried out by persons, who visit the science festivals, open laboratories and science picnics.

References

[1] http://www.tornadotube.net

[2] http://www.ekow.com

[3] D. Halliday, R. Resnick, and J. Walker, *Fundamentals of Physics, Part 2*, John Wiley and Sons, Inc., New York 2001, p. 72.

[4] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics*, Pergamon Press, New York 1975, p. 67.

[5] W. Thirring, *Lehrbuch der Mathematischen Physik, Band 1, Klassische Dynamische Systeme*, Springer-Verlag, Wien, New York 1997, p. 162.

[6] D. Halliday, R. Resnick, and J. Walker, *Fundamentals of Physics, Part 1*, John Wiley and Sons, Inc., New York 2001, p. 162.

[7] F. M. White, *Viscous Fluid Flow*, McGraw-Hill Company, New York 1991, p. 92.

[8] E. Karsley, *Rheology*, in: *Encyclopedia of Applied Physics*, Vol. 16, ed. G. L. Trigg, VCH Publishers Inc., New York-Berlin-Tokyo 1996, p. 465.

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UNIWERSALNA RURA TORNADO

S t r e s z c z e n i e

Opisano budowę przyrzadu złożonego z dwóch plastikowych, przezroczystych butelek, których wyloty połączone są zaworem kulowym, używanym w instalacjach wodociagowych. Po napełnieniu cieczą o małej lepkości przyrząd ten pozwala pokazać wytwarzanie i tłumienie tornada oraz jego samoistne wzbudzanie się. Po wymianie cieczy przyrząd umożliwia badanie ruchu w ośrodku lepkim oraz struktury piany. Gdy przyrząd zostanie napełniony cieczą nienewtonowska można badać efekt Fano.

Słowa kluczowe: wir, butelka, płyn lepki, tornado, piana, efekt Fano

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Stanisław Bednarek and Paweł Tyran

INVESTIGATION OF CORRELATION BETWEEN COEFFICIENTS OF SURFACE TENSION AND VISCOSITY IN FERROFLUIDS

Summary

A stalagmometer placed in Helmholtz coils was constructed together with a capillary viscosimeter, equipped with an exchengeable solenoid and electromagnet. Through these instruments, dependence of the coefficient of surface tension, coefficient of viscosity of the ferrofluid from the induction of magnetic field within range of 0–80 mT were investigated. The examinations were performed in temperatures 15°C and 26°C for different degrees of dilutions of the ferrofluid. High interdependence between values of coefficients of surface tensions and viscosity was detected in both temperatures for magnetic induction directed parallelly and perpendicularly towards direction of the ferrofluid's flow.

Keywords and phrases: ferrofluid, surface tension, viscosity, magnetic field, coupling

1. Introduction

Ferrofluids are suspensions of ferromagnetic or superparamagnetic particles of nanometer dimensions in a dispersive liquid. In order to provide stability of the suspension, the particles are coated with a layer of a surfactant, which prevents aggregation of particles' sedimentation [1, 2]. Because of unique combination of liquid qualities and strong reaction with a magnetic field, ferrofluids found numerous applications, among others in clutches, bearings, measurement instruments or image diagnostics [1, 3]. New applications of ferrofluids, e.g. in printers or cancer therapy are planned [4]. Because of that, careful knowledge of ferrofluids' properties is of significant meaning also for broadening the scope of the materials' application. Results of research on magnetic field's influence on the viscosity of ferrofluids are known as well as the results of separately performed studies over balance of ferrofluids' drops in a magnetic field [5–9]. This article demonstrates results of the research of changes in both viscosity and surface tension of ferrofluids in a magnetic field. The performed research showed strict interdependence of these quantities.

2. Measurement method

The coefficient of the ferrofluid's surface tension was determined through a magnetic stalagmometer of self made construction, Fig. 1, Photo 1. In this instrument, the ferrofluid flow out a vertical tube, equipped with a valve and a capillary. At the lower end of the capillar, drops of the ferrofluid were formed. Speed of the drops formation was controlled by using the valve. The drop detached from the lower end of the capillar, when its dimensions increased to properly high values. For those values, weight of the drop exceeded strengths of the surface tension, influencing the circumference of the narrowest part of the drop, Fig. 2. The tube with the ferrofluid was located in the middle part of the Helmholtz coils system, which enabled production of a homogenous magnetic field, of 0–80 mT induction. Induction value of this field was measured with a hallotron teslameter.

A known method of determining the coefficient of surface tension σ through a stalagmometer, consists in an assumption that upon detachment, the drop is of spherical shape, and in comparison of its weight W with strength of surface tension F_n [10]. Deviation from the spherical shape caused by necking is taken into consideration by introducing a corrective coefficient, whose value depends on the relation of the capillar's diameter to the drop's diameter, and it is provided in the special tables. In the magnetic field, shape of the drop differed significantly from the spherical one, Fig. 3, Photo 2. Because of that, it was necessary to apply own method. It consisted in the fact that creation of the drop was filmed. A photography representing a drop just before it detached was chosen. The photograph was enlarged by 80 times, and on that basis a drawing of a factual size of the drop on an axial plane was prepared. This profile was divided into horizontal stripes, each $\Delta h = 1\,\mathrm{mm}$ wide, and their lengths were measured, Fig. 4. Half of length of every stripe equaled the r_i radius, an elementary cylinder of volume ΔV_i, which the drop was divided into. Results from those measurements were used to calculate a complete volume V of the drop, according to the formula

$$V = \pi \Delta h_i \sum_{i=1}^{n} r_i^2. \tag{1}$$

Strength of the surface tension F_n and weight of the drop W are expressed by the following formulas:

$$F_{nb} = 2\pi r_p \sigma, \tag{2}$$

$$W = \rho g V. \tag{3}$$

On the basis of formulas (1–3) to the condition $F_n = W$, a formula for the coefficient of surface tension σ was received

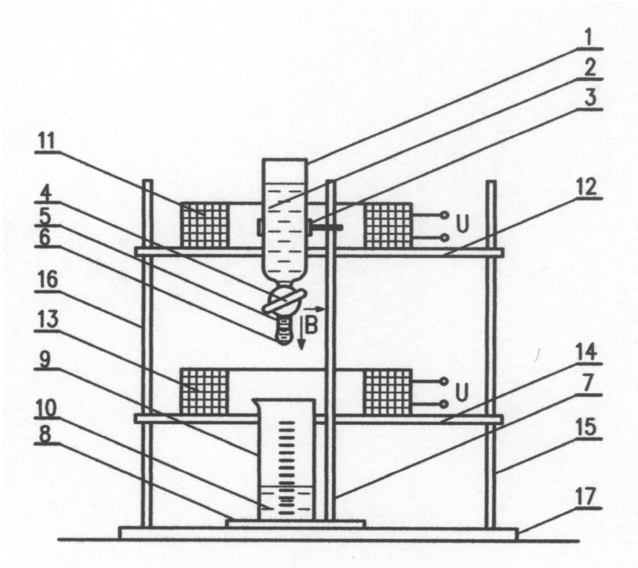

Fig. 1: Structure of a magnetic stalagmometer; 1 – a vertical tube, 2 – ferrofluid in a tube, 3 – a tube's holder, 4 – a valve, 5 – a capillary, 6 – a ferrofluid's drop, 7 – a tube's bracket, 8 – a stalagmometer's base, 9 – a measuring cylinder, 10 – ferrofluid in the measuring cylinder, 11 – upper coil, 12 – base of the upper coil, 13 – lower coil, 14 – base of the lower coil, 15, 16 – coils' supports, 17 – common base of coils, U – voltage of supply, B – induction of the magnetic field.

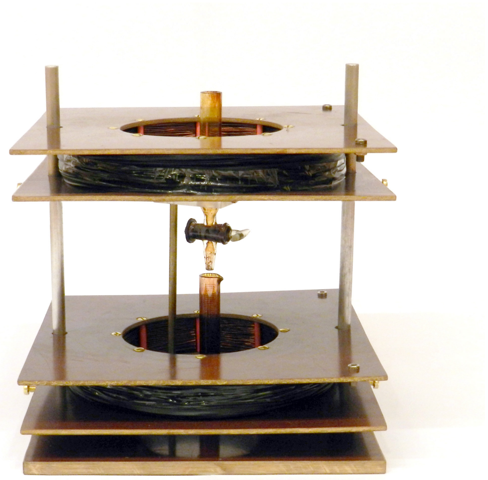

Photo 1: General view of the stalagmometer placed in Helmholtz,s coils.

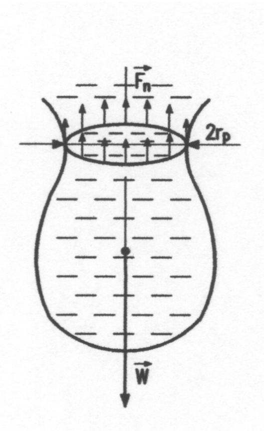

Fig. 2: Forces holding the hanging drop in balance; F_n – force of the surface tension, W – weight of the drop, r_p – necking radius.

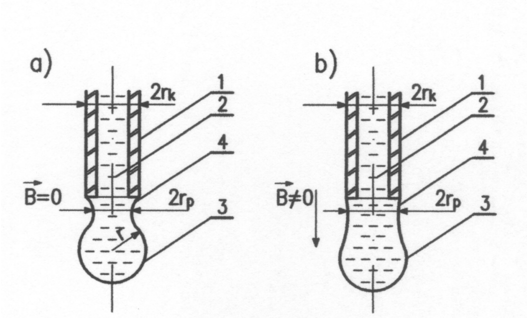

Fig. 3: Shape of the ferrofluid's drop: a) without a magnetic field, b) in a magnetic field; 1 – capillary, 2 – ferrofluid in the tube, 3 – ferrofluid's drop, 4 – necking before the drop's detachment, r_k – external radius of the capillary, r_p – necking radius, r – drop's radius, B – induction of the magnetic field.

Photo 2: An example of change of shape of the ferrofluid's drop in a magnetic field: a) without the magnetic field, b) in the magnetic field of induction 60 mT.

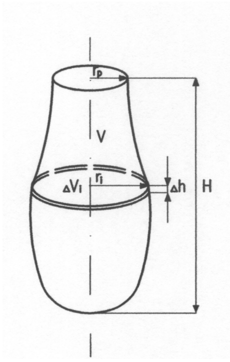

Fig. 4: A method of volume calculation of the ferrofluid's drop; r_i – layer's radius, Δh – layer's height, ΔV_i – layer's volume, r_p – necking radius, H – drop's height, V – drop's volume.

$$\sigma = \frac{\rho g \Delta h}{2r_p} \sum_{i=1}^{n} r_i^2. \tag{4}$$

In the formula (4) g means gravitational acceleration, and ρ means density of the ferrofluid.

In aim to determine coefficients of the ferrofluid's viscosity, capillar viscosimeter of own construction was used, where the magnetic field was applied in a parallel or perpendicular direction towards the direction of the ferrofluid's flow, Figs. 5, 6, Photos 3, 4. Thanks to that, it was possible to determine two coefficients of viscosity η_1, η_2, corresponding with parallel and perpendicular application of the field. A Mariotte's bottle, consisting of a controlling pipe 3 and a cork 4, placed in a vertical tube 1 filled with the ferrofluid 2. It allowed to maintain the agreed value of pressure and speed of the ferrofluid's flow, while its level was above the lower end of the controlling pipe. The flow speed was determined by moving the controlling pipe 3 in the cork 4. Flow of the ferrofluid in the viscosimeter was laminar and it may be described with the Hagen-Poisuille's formula. In a general case, this formula has the following form

$$V = \frac{\pi r^4 \Delta p t}{8 l \eta} = \frac{\Delta p t}{\left(\frac{8 l \eta}{\pi r^4}\right)}. \tag{5}$$

Meaning of the symbols in the formula (5) is the following: V – volume of the flowing fluid, t – time of the flow, Δp – difference of pressures between the ends of the capillary 6, which causing flow of ferrofluid, η – fluid viscosity coefficient, l, r – respectively: length and radius of the tube. Expression $((8l\eta/\pi r^4$ in the formula (5) plays a role of hydraulic resistance. In the constructed viscosimeter there are also two segments of a capillary 6 with l_1, l_2 lengths, along which there is no magnetic field, and ferrofluid's viscosity on that segments equals η_0, Fig. 7. These segments also cause hydraulic resistance, which is why the Hagen-Poisuille's formula for the constructed viscosimeter has the following formula:

$$V = \frac{\Delta p t}{\frac{8 l \eta_{1,2}}{\pi r_w^4} + \frac{8 l (l_1 + l_2) \eta_0}{\pi r_w^4}}. \tag{6}$$

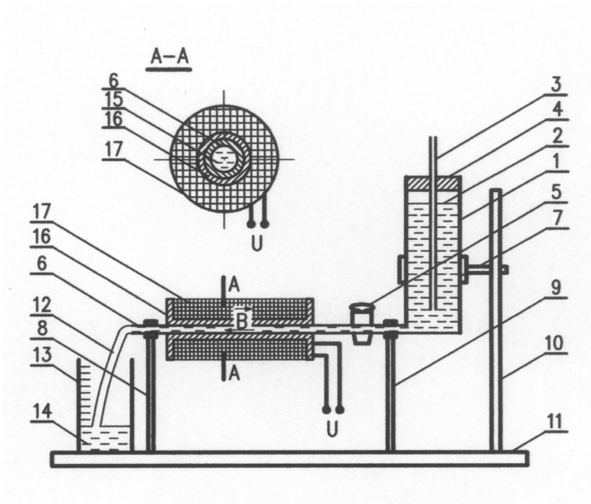

Fig. 5: Structure of a capillary viscosimeter with longitudinal magnetic field and a Mariotte's bottle; 1 – vertical tube, 2 – ferrofluid in the tube, 3 – controlling pipe, 4 – cork, 5 – valve, 6 – capillary, 7 – holder of the vertical tube, 8, 9, 10 – brackets, 11 – base, 12 – ferrofluid's flow, 13 – measuring cylinder, 14 – ferrofluid in the measuring cylinder, 15 – ferrofluid in the capillary, 16 – spool of solenoid, 16 – winding of the solenoid, U – voltage of supply, B – induction of the magnetic field.

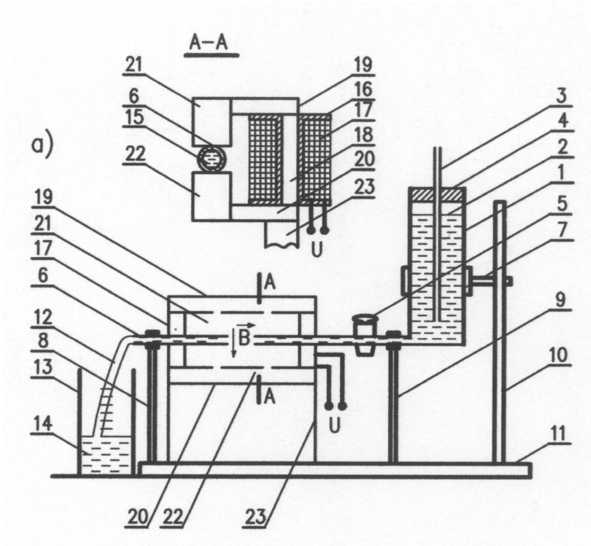

Fig. 6: Structure of a capillary viscosimeter with a transversal magnetic field and a Mariotte's bottle; numbers 1–15 – are of the same meaning as in the Fig. 5 caption, 16 – frame of coils, 17 – electromagnet's winding, 18 – electromagnet's core, 19, 20 – horizontal pole shoes, 21, 22 – vertical pole shoes, 23 – electromagnet's support, U – voltage of supply, B – induction of the magnetic field.

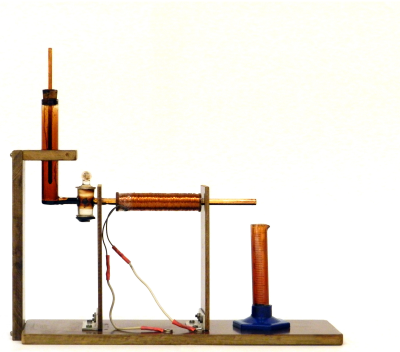

Photo 3: General view of the viscosimeter with longitudinal magnetic field and a Mariotte's bottle.

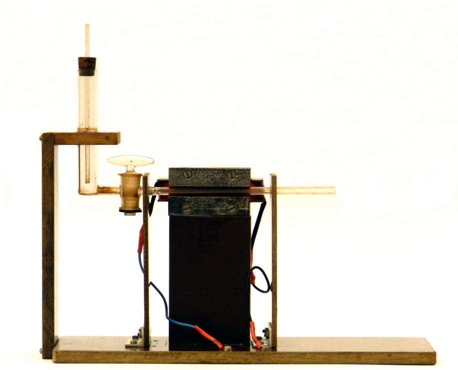

Photo 4: General view of the viscosimeter with transversal magnetic field and a Mariotte's bottle.

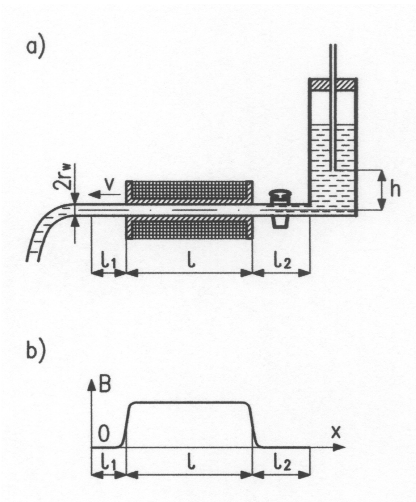

Fig. 7: Parameters of a capillary viscosimeter with magnetic field and a Mariotte's bottle: a) basic dimensions, b) spatial distribution of induction of the magnetic field along the capillary; h – effective height of the ferrofluid's column, r_w – internal radius of the capillary, l – length of the segment of the capillary with an effective magnetic field, l_1, l_2 – lengths of segments of the capillary without the magnetic field, B – induction of the magnetic field, v – velocity of flow, 0x – coordinate axis.

In the formula (6) $\eta_{1,2}$, mean the viscosity coefficient of the ferrofluid for a magnetic field directed, respectively: in parallel or perpendicularly to the direction of the flow, r_w – internal radius of the capillary 6, where the ferrofluid flows. To simplify further discussion, η_1 will be called as a coefficient of longitudinal viscosity, and η_2 – as a coefficient of transversal viscosity. The difference of pressures, causing flow of the ferrofluid is expressed with the formula

$$\Delta p = \rho g h, \tag{7}$$

where h means a distance from the lower end of the controlling tube 3 to the axis of the capillary 6. After substituting the formula (6) to (5) and transformation, the following formula for the ferrofluid's viscosity coefficient in a magnetic field were obtained:

$$\eta_{1,2} = \frac{\pi r^4 \rho g h t}{8lV} - \left(\frac{l_1 + l_2}{l}\right)\eta_0. \tag{8}$$

The formulas (4) and (8) were applied to determine coefficients of surface tension and the ferrofluid's viscosity.

For the research, a ferrofluid containing of Fe_3O_4 magnetite's particles with dimensions of 10–20 nm, obtained by a method of chemical polycondensation was applied. This method consists in production of particles from a mixture of iron chloride solutions with different valences in a result of their alkalization [2]. The produced particles were dispersed in mineral oil. A surfactant, preventing agglomeration of particles was oleic acid. Content of iron in the ferrofluid constituted 6.15% of its weight, and it was determined with a method of an X-ray microanalysis. Density of the ferrofluid was $1.21\,\mathrm{g/cm^3}$. In the research the ferrofluid without dilution and solutions of the ferrofluid obtained through adding 30% or 60% of the diluter – a mixture of toluene and acetone – was used. The volume proportion of toluene and acetone in the diluter was 7 : 3. All investigation were made in the magnetic field with the induction changed within the range of 0–80 mT and two temperatures 15°C and 26°C. The results obtained in described investigations are presented on Figs. 8–13.

3. Discussion of results

In all series of the measurements, significant increases of coefficients of viscosity and surface tensions were detected in the investigated ferrofluids, caused with a growth of induction of the magnetic field within the range of 0–80 mT. For example, in the temperature of 26°C, coefficient of the surface tension of the undiluted ferrofluid, without a magnetic field was $38 \pm 2\,\mathrm{mN/m}$, and in the magnetic field with induction of 80 mT, it increased to $75 \pm 4\,\mathrm{mN/m}$, i.e. 1.97 times, Fig. 8. The coefficient of longitudinal and transversal viscosity without the magnetic field, in the same circumstances was, respectively $126 \pm 7\,\mathrm{mPa \cdot s}$ and $124 \pm 7\,\mathrm{mPa{\cdot}s}$, Figs. 10, 12. It means equality of those coefficients within boundaries of an measurement error, and lack of anisotropy of the ferrofluid's viscosity.

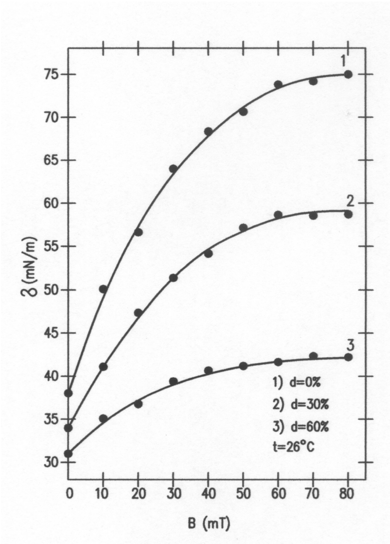

Fig. 8: Dependence of coefficients of surface tension σ of the ferrofluid on induction of the magnetic field B for various concentration levels d in the temperature $t = 15°$C.

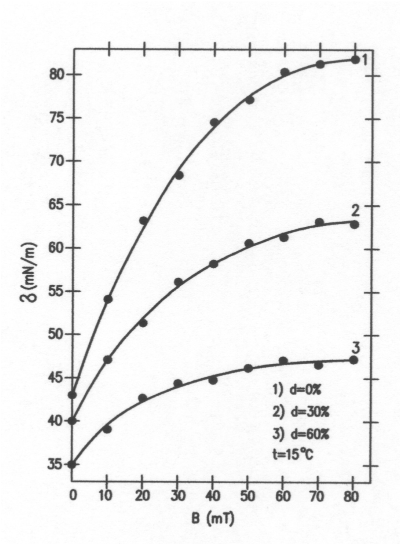

Fig. 9: Dependence of coefficients of surface tension σ of the ferrofluid on induction of the magnetic field B for various concentration levels d in the temperature $t = 26°$C.

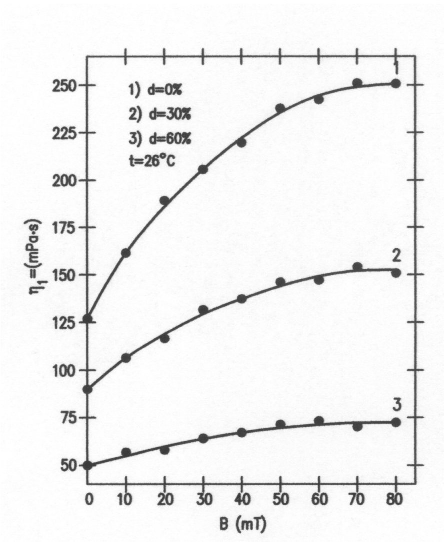

Fig. 10: Dependence of horizontal viscosity coefficients η_1 of ferrofluid on induction of the magnetic field B for different concentration levels d in temperature $t = 15°$C.

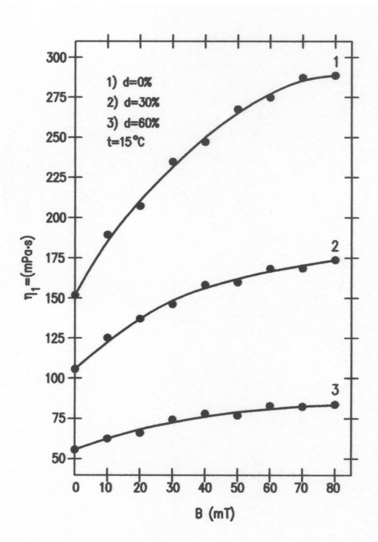

Fig. 11: Dependence of horizontal viscosity coefficients η_1 of ferrofluid on induction of the magnetic field B for different concentration levels d in temperature $t = 26°$C.

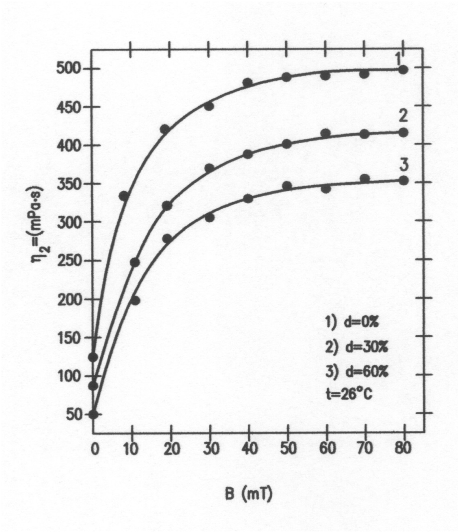

Fig. 12: Dependence of horizontal viscosity coefficients η_2 of ferrofluid on induction of the magnetic field B for different concentration levels d in temperature $t = 15°$C.

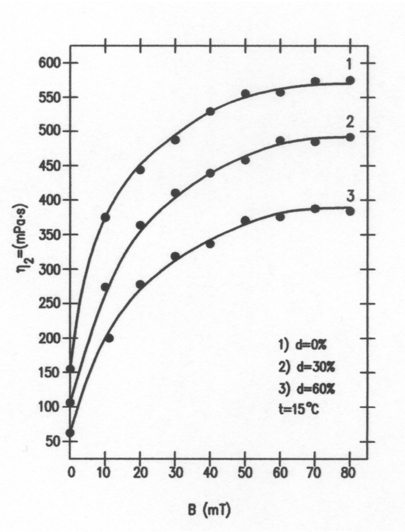

Fig. 13: Dependence of horizontal viscosity coefficients η_2 of ferrofluid on induction of the magnetic field B for different concentration levels d in temperature $t = 26°$C.

After applying a magnetic field with induction of 80 mT, coefficient of the longitudinal viscosity was 250 ± 14 mPa·s, and the coefficient of the transversal viscosity was 500 ± 25 mPa·s, Figs. 10, 12. This mean an increase by 2 and 4.2 times, and significant anisotropy of the ferrofluid's viscosity in the magnetic field. The anisotropy consists in higher increasing of the transversal viscosity coefficient than the longitudinal viscosity coefficient for the same increasing of the magnetic field's induction. The increase of the ferrofluid's viscosity coefficient in the magnetic field was detected previously by other researchers, and results of this work are similar with previous ones [11–14].

Dependencies of coefficients of surface tension and viscosity from induction of the magnetic field shows of nonlinear character. This nonlinearity consists in the fact that in the initial range of the magnetic field's induction of 0–20 mT, increases of these coefficients are higher than in the final range of induction changes 60–80 mT. The nonlinear dependencies of coefficients are analogical with curves that present dependency of magnetization of a ferromagnetic or superparamagnic materials on induction of the applied magnetic field. Lower increases of coefficient in the final ranges of induction changes are caused by the fact that for induction of the magnetic field in the range 100–200 mT, magnetization of ferrofluids achieve a saturation level [1, 2].

Addition of a solvent caused a decrease in coefficients of both surface tension and viscosity. For example, after adding a 60% solvent in the temperature of 26°C, the coefficient of surface tensions decreased from $38 \pm 2\,\mathrm{mN/m}$ to $31 \pm 2\,\mathrm{mN/m}$, Fig. 7. The longitudinal and transversal viscosity coefficients decreased as a result of dilution as follows: $126 \pm 7\,\mathrm{mPa{\cdot}s}$ to $50 \pm 7\,\mathrm{mPa{\cdot}s}$ and $124 \pm 7\,\mathrm{mPa{\cdot}s}$ also to $50 \pm 7\,\mathrm{mPa{\cdot}s}$, and were equal within limits of a measurement error, Figs. 10, 12. These results shows conservation of anisotropy of the ferrofluid's viscosity without the magnetic field, also after its dilution. The detected changes are caused by the fact that the diluter has a lower surface tension and viscosity coefficients than the ferrofluid [15]. Lowering proper coefficients of the ferrofluid was proportional to the diluter's content. The detected changes comply with expectations because in the mixture of fluids that chemically do not react with each other, the mixture's viscosity coefficient is directly proportional to the components and their coefficients of viscosity [16]. These results are also analogical to the results from previous investigation [17–19].

The research repeated in the temperature of 15°C showed than lowering temperature will cause an increase of the surface tension coefficient and coefficients of longitudinal and transversal viscosity of all ferrofluids. For example, a coefficient of surface tension of a non-diluted ferrofluid increased from $38 \pm 2\,\mathrm{mN/m}$ to $43 \pm 2\,\mathrm{mN/m}$, and the coefficient of longitudinal and transversal viscosity increased respectively from $126 \pm 7\,\mathrm{mPa{\cdot}s}$ to $152 \pm 8\,\mathrm{mPa{\cdot}s}$ and from $124 \pm 7\,\mathrm{mPa{\cdot}s}$ to $152 \pm 8\,\mathrm{mPa{\cdot}s}$, Figs. 8–13. The provided values refer to a situation without a magnetic field. After applying the magnetic field, similar increases of proper values took place. These results also comply with expectations, because termodynamics' predictions and results of other investigations suggest that coefficients of surface tension and viscosity decrease together with an increase of temperature. Value of the surface tensions coefficient equal zero in the critical temperature, when the difference between a liquid and a gas phase disappears [16, 18].

The most significant and a new result of the made investigations is detection of coupling of surface tension and viscosity coefficients. Obtained curves presenting that the dependence of surface tension and viscosity coefficients shows very similar shape. In order to compare them, coefficients of linear correlation were calculated between series of values of both coefficients. The obtained values of correlation coefficients are collected in Tab. 1 and 2. All obtained values of correlation coefficients are significant on the level of confidence $\alpha = 0.95$. A reason for these correlation lays in changes of the ferrofluid's structure caused by influence of the applied magnetic field to the magnetite's particles. Before application of the magnetic field, spatial distribution of particles in the ferrofluid was disordered, Fig. 14. In such a situation the ferrofluid acted similarly to Newtonian fluids. The ferrofluid's flow took place with relatively small resistances to motion. The ferrofluid's viscosity coefficient is permanent then and it is expressed with the Einstein's formula

$$\eta_f = \eta\left(1 + \frac{5}{2}\varphi\right), \tag{9}$$

where η_f – ferrofluid's viscosity, η – dispersive liquid's viscosity, φ – a filling factor of the ferrofluid's volume by particles ($\varphi \ll 1$) [20].

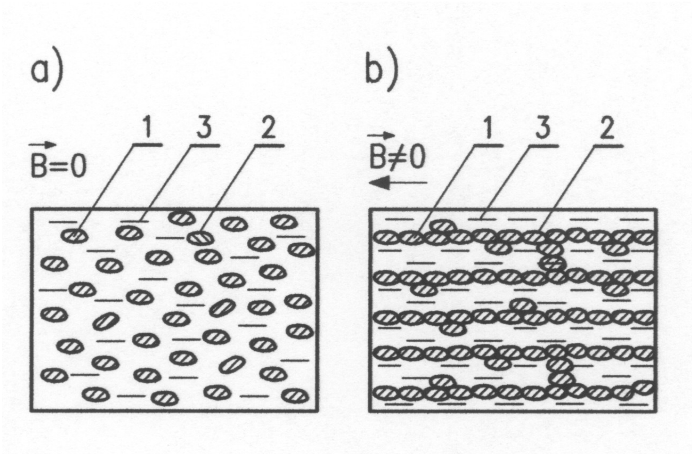

Fig. 14: Ferrofluid's structure: a) without magnetic field, b) after applying the magnetic field; 1 – magnetite's particle, 2 – a surfactant, 3 – a dispersion liquid, B – induction of the magnetic field.

Tab. 1: Set of correlation coefficients r between coefficients of surface tension σ and longitudinal and transversal viscosity coefficients η_1, η_2 of a ferrofluid with various stages of dilution d in the temperature $t = 15^\circ$C, critical value of the correlation coefficient $r_{\alpha k}$ is 0.62 at the confidence level $\alpha = 0.95$.

Dilution d(%)	Coefficient of correlation r between:			
	σ and η_1		σ and η_2	
	value	significance	value	significance
0	0.96	yes	0.95	yes
30	0.94	yes	0.93	yes
60	0.97	yes	0.95	yes

Tab. 2: Set of correlation coefficients r between coefficients of surface tension σ and longitudinal and transversal viscosity coefficients η_1, η_2 of a ferrofluid with various stages of dilution d in the temperature $t = 26^\circ$C, critical value of the correlation coefficient $r_{\alpha k}$ is 0.62 at the confidence level $\alpha = 0.95$.

Dilution d(%)	Coefficient of correlation r between:			
	σ and η_1		σ and η_2	
	value	significance	value	significance
0	0.92	yes	0.97	yes
30	0.94	yes	0.93	yes
60	0.93	yes	0.95	yes

Application of the magnetic field led to magnetization of particles and induction of dipole moments within them. Mutual interaction of those dipoles caused rotations and shifts of the particles. As a result, the particles approached each other in a way that opposite poles of dipoles was possibly close to each other, and the homogeneous poles were as far as possible. As a result, the particles were grouped in chains directed along the direction of the applied magnetic field. Between some chains present close to each other, so called bridges could be formed of diagonally or perpendicularly directed chains of particles. In this manner is created a more ordered chain's structure of the ferrofluid, described for the first time by Winslow [21]. This structure also results from the particles' distribution tend to minimum of free energy in the magnetic field.

After forming an ordered structure, the ferrofluid's flow is significantly slower, because its motion is inhibited through chains of particles interacting with each other. To cause the ferrofluid's movement along the magnetic field, it is necessary to translate these chains towards each other. In turn, to cause the ferrofluid flow perpendicularly to the magnetic field, it is mainly necessary to shear the chains of particles. In the second case, influence of greater forces is indispensable. Therefore, the transversal viscosity coefficient is higher than the longitudinal viscosity coefficient. What is more, chains of particles directed along the magnetic field may hold a bigger ferrofluid's drop. These chains appear not only on the surface of the hanging drop but in its whole cross-section. Before detachment of the drop it is necessary to disrupt these chains. It requires properly significant force, hence an increase of the ferrofluid's surface tension coefficient was observed, Fig. 15.

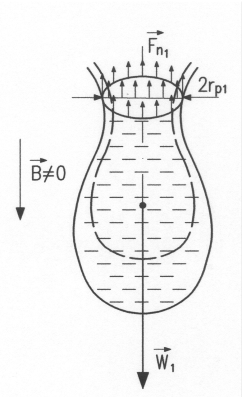

Fig. 15: Influence of the magnetic field on the change of the surface tension coefficient; F_{n1} – force of the surface tension, W_1 – weight of the drop, r_{p1} – necking radius, the dashed line marks shape of the drop before applying the field, B – induction of the magnetic field.

A conclusion drawn here that in order to cause flow of the ferrofluid and to generate the drop it is necessary to disrupt the particles' chains. Both phenomena have a common cause, which is why a strict correlation between the viscosity coefficients and ferrofluid's surface tension was observed. Furthermore, the process of creation, shape and size of the ferrofluid's drop placed in the magnetic field undergo not only to interaction of the particles of the dispersion fluid and magnetite's particles on the ferrofluid's surface. Influence of chains of magnetite's particles on the surface of the whole drop's cross-section is important. This situation significantly differentiates the process of generation of the ferrofluid's drop in the magnetic field from the generation process of drops of other fluids or the ferrofluid's drop without the magnetic field. Results of the made investigations also suggest that there is a need to define the coefficient of surface tension of a ferrofluid in a magnetic field in a different manner. This definition should also take into account influence of the column's cross-section field or the ferrofluid's layer.

References

[1] K. Raj, B. Moskowitz, and R. Casciari, *Advances in ferrofluid technology*, J. Mag. Mag. Mater. **149** (1995), 174–180.

[2] E. Blums, P. Ceber, and M. M. Maiorov, *Magnetic Fluids*, ed. W. de Gruyter, Berlin-New York 1997.

[3] J. M. Ginder, *Rheology controlled by magnetic fields*, in: *Encyclopedia of Applied Physics*, ed. G. Trigg, New York-Berlin-Cambridge-Tokyo **16** (1996), 487–503.

[4] C. Aleixou, R. Tietze, E. Schreiber, R. Jurgous, H. Richter, L. Trahms, H. Rahn, S. Oldenbach, and S. Lyer, *Cancer therapy with loaded magnetic nanoparticles – magnetic drug targeting*, Proceedings of the 12th International Conference on Magnetic Fluids, Sendai, Japan 1–5 August 2010; J. Mag. Mag. Mater. **323** (2011), 1404–1407.

[5] J. C. Bacri and D. Salin, *Instability of ferrofluids magnetic drop under magnetic field*, J. Phys. Letters **43**, no. 17 (1982), 649–654.

[6] V. I. Drozdova and T. V. Skrobotova, *On deformation of magnetizable drops in magnetic field*, Materials 2nd All-Union School-Seminary of Ferrofluids, Moscow 1981, 24–25.

[7] D. K. Wagh and A. Avashia, *On the viscosity of a magnetic fluid*, J. Mag. Mag. Mater. **153** (1996), 359–365.

[8] J. P. McTague, *Magnetoviscosity of the magnetic colloids*, J. Chem. Phys. **51**, no. 1 (1969), 136–139.

[9] W. F. Hall and S. N. Busenberg, *Viscosity of magnetic suspension*, J. Chem. Phys. **51**, no. 1 (1969), 137–144.

[10] F. Kohlraush, *Praktische Physik*, Band I, Verlagsgesellschaft, Stuttgart 1955, p. 240.

[11] J. C. Bacri, J. Dumas, D. Grise, R. Perzynski, and D. Salin, *Ferrofluid viscometer*, J. Phys. Letters **46**, no. 24 (1983), L 1119–1205.

[12] Y. N. Mozgovoi and E. Blums, *The flow of ferromagnetic liquid in a magnetic field*, Magnetohydrodynamics **19**, no. 1 (1973), 52–57.

[13] A. C. Levi, R. F. Hobson, and F. R. McCourt, *Magnetoviscosity of colloidal suspension*, Can. J. Phys. **51**, no. 2 (1973), 180–194.

[14] T. Weser and K. Stiestadt, *Magnetoviscosity of concentrated ferrofluids*, Zeitschrift der Physik B, **18**, no. 12 (1985), 2325–2341.

[15] A. M. James and M. P. Lord, *Macmillan's Chemical and Physical Data*, The Macmillan Press Ltd., London-Basingstoke 1992, p. 27.

[16] W. Demtröder, *Experimentalphysik 1, Mechanik und Wärme*, Springer, Berlin 2007, p. 219.

[17] H. Brenner, *Rheology of a dilute suspension of dipolar spherical particles in an external field*, J. Colloid. Interface Sci. **32**, no. 1 (1970), 141–158.

[18] H. Brenner and M. H. Weissman, *Rheology of a dilute suspension of dipolar spherical particles in an external field. Effect of rotary Brownian motion*, J. Colloid. Interface Sci. **41**, no. 3 (1972), 499–531.

[19] H. Brenner, *Rheology of a dilute suspension of axisymmetric Brownian particles*, Int. J.Multiphas. Flow **1**, no. 2 (1974), 195–204.

[20] K. H. Lautenschläger, W. Schröter, and A. Wanninger, *Taschenbuch der Chemie*, Auflage 20, Wissenschaftlicher Verlag Harri Deutsch GmbH, Frankfurt am Main 2005, p. 278.

[21] W. M. Winslow, *Induced fibration of suspension*, J. Appl. Phys. **20**, no. 12 (1949), 1137–1140.

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BADANIE KORELACJI MIĘDZY WSPÓŁCZYNNIKAMI NAPIĘCIA POWIERZCHNIOWEGO I LEPKOŚCI FERROFLUIDÓW

S t r e s z c z e n i e

Zbudowano stalagmometr umieszczony w cewkach Helmholtza oraz wiskozymetr przepływowy, wyposażony w wymienny solenoid i elektromagnes. Za pomocą tych przyrządów zbadano zależność współczynnika napięcia powierzchniowego i współczynnika lepkości ferrofluidu od indukcji pola magnetycznego w zakresie 0–80 mT. Badania przeprowadzono w temperaturach 15°C i 26°C dla różnych stopni rozcieńczenia ferrofluidu. Wykryto wysoką współzależność między wartościami współczynników napięcia powierzchniowego i lepkości w obu temperaturach dla indukcji magnetycznej skierowanej równolegle, jak również prostopadle do kierunku przepływu ferrofluidu.

Słowa kluczowe: ferrofluid, napięcie powierzchniowe, lepkość, pole magnetyczne

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DIFFERENTIAL EQUATIONS FOR SECANTOPTICS

Summary

The set of intersection points of appropriate pair of secants to a given oval C form a curve which we called a secantoptic $C_{\alpha,\beta,\gamma}$ of an oval C. In [14], [9], [10] and [11] we proved many properties of secantoptics and we found relations between notions of secantoptics, evolutoids, hedgehogs and isoptics of a pair of curves. In this paper we present some differential equations connected to secantoptics.

Keywords and phrases: isoptic curve, secantoptic, envelope

1. Introduction

In this paper a plane, closed, simple, strictly convex, smooth curve C we call an oval. Let us introduce a coordinate system with the origin O in the interior of C and denote by $p(t)$, where $t \in [0, 2\pi)$, the support function of the curve C. As it was shown in [13] this support function is differentiable and C can be parametrized by

$$z(t) = p(t)e^{it} + \dot{p}(t)ie^{it} \quad \text{for} \quad t \in [0, 2\pi). \tag{1.1}$$

Note that for any oval C we have $p(t) + \ddot{p}(t) > 0$ for $t \in [0, 2\pi)$ and the expression $R(t) = p(t) + \ddot{p}(t)$ is the radius of curvature of C.

Considered notion of a secantoptic is a generalization of the notion of isoptic curve known since 18th century. For α fixed in the interval $(0, \pi)$ the isoptic C_α of C is a set of points from which the oval C is seen under the angle α. The equation of C_α is (see [3])

$$z_\alpha(t) = p(t)e^{it} + \{-p(t)\cot\alpha + \frac{1}{\sin\alpha}p(t+\alpha)\}ie^{it}, \ t \in [0, 2\pi), \tag{1.2}$$

where $p(t)$ denotes the support function of C. Let us recall the notion of a secantoptic.

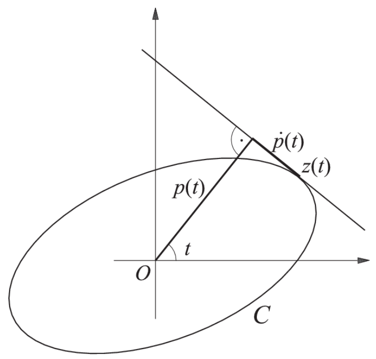

Fig. 1: A parametrization of a convex curve with a support function.

Let C be an oval and let

$$\beta \in [0, \pi), \quad \gamma \in [0, \pi - \beta) \quad \text{and} \quad \alpha \in (\beta + \gamma, \pi)$$

be fixed angles. By $l_1(t)$ we denote the tangent to C at $z(t)$. Let us rotate $l_1(t)$ arround $z(t)$ through an angle $-\beta$. Such obtained secant of C we denote by $s_1(t)$. Similarly let $l_2(t) = l_1(t + \alpha - \beta - \gamma)$ be the tangent to C at $z(t + \alpha - \beta - \gamma)$. By $s_2(t)$ we denote the secant to C obtained by rotation of $l_2(t)$ arround the tangency point through an angle γ. Then $s_1(t)$ and $s_2(t)$ intersect forming the angle α.

Definition 1.1. *The set of points $z_{\alpha,\beta,\gamma}(t)$ of intersection of secants $s_1(t)$ and $s_2(t)$ to the oval C for $t \in [0, 2\pi]$ forms a curve $C_{\alpha,\beta,\gamma}$ which we call a secantoptic of an oval C.*

If we introduce the following notation

$$\begin{aligned} q(t) &= z(t) - z(t + \alpha - \beta - \gamma), \\ b(t) &= [q(t), e^{it}], \\ B(t) &= [q(t), ie^{it}], \\ q(t) &= (B(t) - ib(t))e^{it}, \end{aligned}$$

$$\lambda(t) = \frac{b(t)\sin(\alpha - \beta) - B(t)\cos(\alpha - \beta)}{\sin\alpha}, \tag{1.3}$$

$$\mu(t) = -\frac{b(t)\sin\beta + B(t)\cos\beta}{\sin\alpha}, \tag{1.4}$$

where $[v, w] = ad - bc$ for $v = a + bi$ and $w = c + di$, then the parametrization of a secantoptic $C_{\alpha,\beta,\gamma}$ of an oval C can be written as

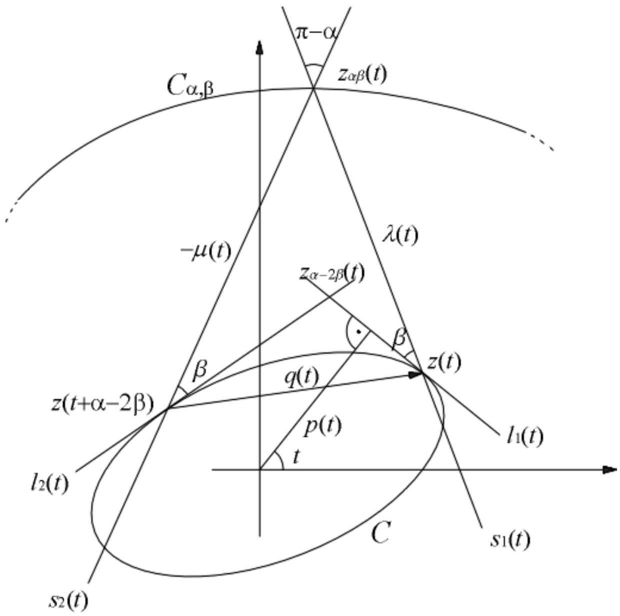

Fig. 2: Construction of a secantoptic of an oval.

$$z_{\alpha,\beta,\gamma}(t) = (p(t) + \lambda(t)\sin\beta + i(\dot{p}(t) + \lambda(t)\cos\beta))e^{it} \quad \text{for} \quad t \in [0, 2\pi).$$

Let us notice that

$$\lambda(t) = \frac{1}{\sin\alpha}(-p(t)\cos(\alpha-\beta) - \dot{p}(t)\sin(\alpha-\beta) + p(t+\alpha-\beta-\gamma)\cos\gamma + \\ + \dot{p}(t+\alpha-\beta-\gamma)\sin\gamma), \tag{1.5}$$

$$\mu(t) = \frac{-1}{\sin\alpha}(p(t)\cos\beta - \dot{p}(t)\sin\beta - p(t+\alpha-\beta-\gamma)\cos(\alpha-\gamma) + \\ + \dot{p}(t+\alpha-\beta-\gamma)\sin(\alpha-\gamma)). \tag{1.6}$$

We can compute

$$\frac{\partial b(\alpha,t)}{\partial\alpha} = R(t+\alpha-\beta-\gamma)\cos(\alpha-\beta-\gamma), \tag{1.7}$$

$$\frac{\partial B(\alpha,t)}{\partial\alpha} = R(t+\alpha-\beta-\gamma)\sin(\alpha-\beta-\gamma), \tag{1.8}$$

$$\frac{\partial\lambda(\alpha,t)}{\partial\alpha} = \frac{1}{\sin\alpha}(R(t+\alpha-\beta-\gamma)\sin\gamma - \mu(t)). \tag{1.9}$$

$$\frac{\partial\lambda(\alpha,t)}{\partial t} = \frac{1}{\sin\alpha}(b(t)\cos(\alpha-\beta)+B(t)\sin(\alpha-\beta)-R(t)\sin(\alpha-\beta)+ \tag{1.10}$$
$$+ R(t+\alpha-\beta-\gamma)\sin\gamma).$$

2. Differential equations

In this section we will prove some differential equations connected to secantoptics which are a generalization of equations derived for isoptics in [4]. Obtained results can be useful in further studies on evolutions of plane convex curves. The evolutions of plane curves were examined for example in papers [2], [5], [6], [7], [8] and [12].

Theorem 2.1. *Let C be an oval with a support function $p(t)$. Let*

$$t \mapsto z_{\alpha,\beta,\gamma}(t) = (p(t)+\lambda(\alpha,t)\sin\beta + i(\dot p(t)+\lambda(\alpha,t)\cos\beta))e^{it} \tag{2.1}$$

be a secantoptic of an oval C. Then the function $\lambda(\alpha,t)$ satisfy the following equation

$$\frac{\partial\lambda}{\partial\alpha} - \frac{\partial\lambda}{\partial t} + \lambda(\alpha,t)\cot\alpha = R(t)\frac{\sin(\alpha-\beta)}{\sin\alpha}. \tag{2.2}$$

Moreover, $\lambda(\beta+\gamma,t)=0$ and $\lambda(-,t)$ is increasing function.

Proof. Using formulas (1.9) and (1.11) we have

$$\frac{\partial\lambda}{\partial\alpha} - \frac{\partial\lambda}{\partial t} = -\lambda(\alpha,t)\frac{\cos\alpha}{\sin\alpha} + R(t)\frac{\sin(\alpha-\beta)}{\sin\alpha},$$

hence the equation (2.2) is satisfied. If $\beta+\gamma\neq 0$, then from the formula (1.5) we get

$$\lambda(\beta+\gamma,t) = \frac{1}{\sin(\beta+\gamma)}(-p(t)\cos\gamma - \dot p(t)\sin\gamma + \dot p(t)\sin\gamma + p(t)\cos\gamma) = 0.$$

If $\beta+\gamma=0$, then $C_{\alpha,\beta,\gamma}$ is the isoptic. For isoptics it is known, see [4], that $\lambda(0,t)=0$. The function $\lambda(-,t)$ is increasing if and only if $\frac{\partial\lambda}{\partial\alpha}>0$, what we will show. Since

$$\frac{\partial\lambda}{\partial\alpha} = \frac{1}{\sin\alpha}(R(t+\alpha-\beta-\gamma)\sin\gamma - \mu(t)) = \frac{-M(t)}{\sin\alpha} > 0$$

from the construction of secantoptics of ovals, then $\lambda(-,t)$ is increasing function. □

Theorem 2.2. *Let C be an oval with a support function $p(t)$. Let*

$$t \mapsto z_{\alpha,\beta,\gamma}(t) = (p(t)+\lambda(\alpha,t)\sin\beta + i(\dot p(t)+\lambda(\alpha,t)\cos\beta))e^{it}$$

be a secantoptic of an oval C. Then the function $\mu(\alpha,t)$ satisfy the following equation

$$\frac{\partial\mu}{\partial\alpha} + \mu(\alpha,t)\cot\alpha = -R(t+\alpha-\beta-\gamma)\frac{\sin(\alpha-\gamma)}{\sin\alpha}. \tag{2.3}$$

Moreover, $\mu(\beta+\gamma,t)=0$.

Proof. Using formulas (1.4), (1.7) and (1.8) we get

$$\frac{\partial\mu}{\partial\alpha} = \frac{1}{\sin\alpha}(-\mu(\alpha,t)\cos\alpha - R(t+\alpha-\beta-\gamma)\sin(\alpha-\gamma)),$$

what proves the equation (2.3). If $\beta+\gamma\neq 0$, then from the formula (1.6) we get

$$\mu(\beta+\gamma,t) = \frac{1}{\sin(\beta+\gamma)}(-p(t)\cos\gamma + \dot{p}(t)\sin\gamma - \dot{p}(t)\sin\gamma + p(t)\cos\gamma) = 0.$$

If $\beta+\gamma = 0$, then to the limit $\lim_{\alpha\to(\beta+\gamma)^+}\mu(\alpha,t)$ we apply de l'Hospital's rule and we obtain

$$\begin{aligned}\lim_{\alpha\to(\beta+\gamma)^+}\Big[&- \frac{1}{\cos\alpha}(-\dot{p}(t+\alpha-\beta-\gamma)\cos(\alpha-\gamma) + p(t+\alpha-\beta-\gamma)\sin(\alpha-\gamma) + \\ &+ \ddot{p}(t+\alpha-\beta-\gamma)\sin(\alpha-\gamma) + \dot{p}(t+\alpha-\beta-\gamma)\cos(\alpha-\gamma))\Big] = \\ &= \lim_{\alpha\to(\beta+\gamma)^+}\Big[-\frac{1}{\cos\alpha}R(t+\alpha-\beta-\gamma)\sin(\alpha-\gamma)\Big] = \\ &= -\frac{R(t)\sin\beta}{\cos(\beta+\gamma)} = 0,\end{aligned}$$

because with our assumptions $\beta+\gamma=0$ if and only if $\beta=0$ and $\gamma=0$. Then we have $\sin\beta=0$ and from de l'Hospital's rule

$$\lim_{\alpha\to(\beta+\gamma)}\mu(\alpha,t) = 0.$$

□

Theorem 2.3. *Let C be an oval with a support function $p(t)$. Let*

$$t\mapsto z_{\alpha,\beta,\gamma}(t) = (p(t)+\lambda(\alpha,t)\sin\beta + i(\dot{p}(t)+\lambda(\alpha,t)\cos\beta))e^{it}$$

be a secantoptic of an oval C. Then the function $L(\alpha,t)=\lambda(\alpha,t)+R(t)\sin\beta$ satisfy the following equation

$$\frac{\partial L}{\partial\alpha} - \frac{\partial L}{\partial t} + L(\alpha,t)\cot\alpha = R(t)\cos\beta - \dot{R}(t)\sin\beta. \tag{2.4}$$

Proof. Let us notice, that

$$\begin{aligned}\frac{\partial L}{\partial t} &= \frac{\partial\lambda}{\partial t} + \dot{R}(t)\sin\beta, \\ \frac{\partial L}{\partial\alpha} &= \frac{\partial\lambda}{\partial\alpha},\end{aligned}$$

and

$$\frac{\partial L}{\partial\alpha} - \frac{\partial L}{\partial t} = \frac{\partial\lambda}{\partial\alpha} - \frac{\partial\lambda}{\partial t} - \dot{R}(t)\sin\beta = R(t)\frac{\sin(\alpha-\beta)}{\sin\alpha} - \lambda(\alpha,t)\cot\alpha - \dot{R}(t)\sin\beta.$$

Hence we have

$$\frac{\partial L}{\partial\alpha} - \frac{\partial L}{\partial t} + L(\alpha,t)\cot\alpha = R(t)\cos\beta - \dot{R}(t)\sin\beta.$$

□

Theorem 2.4. *Let C be an oval with a support function $p(t)$. Let*

$$t \mapsto z_{\alpha,\beta,\gamma}(t) = (p(t) + \lambda(\alpha,t)\sin\beta + i(\dot{p}(t) + \lambda(\alpha,t)\cos\beta))e^{it}$$

be a secantoptic of an oval C. Then the function

$$M(\alpha,t) = \mu(\alpha,t) - R(t+\alpha-\beta-\gamma)\sin\gamma$$

satisfy the following equation

$$\frac{\partial M}{\partial\alpha} + M(\alpha,t)\cot\alpha = -R(t+\alpha-\beta-\gamma)\cos\gamma - \dot{R}(t+\alpha-\beta-\gamma)\sin\gamma. \tag{2.5}$$

Proof. Let us derivate the function $M(\alpha,t)$ with respect α

$$\begin{aligned}
\frac{\partial M}{\partial\alpha} &= \frac{\partial\mu}{\partial\alpha} - \dot{R}(t+\alpha-\beta-\gamma) = \\
&= \frac{\cos\alpha}{\sin\alpha}(-\mu(\alpha,t) + R(t+\alpha-\beta-\gamma)\sin\gamma) - R(t+\alpha-\beta-\gamma)\cos\gamma - \\
&- \dot{R}(t+\alpha-\beta-\gamma),
\end{aligned}$$

hence we have

$$\frac{\partial M}{\partial\alpha} + M(\alpha,t)\cot\alpha = -R(t+\alpha-\beta-\gamma)\cos\gamma - \dot{R}(t+\alpha-\beta-\gamma)\sin\gamma.$$

□

We are looking for the equation for the function $v = |q| = \sqrt{b^2+B^2}$. Note that for secantoptics are satisfied the following equalities

$$\frac{\partial b}{\partial t} = \frac{\partial b}{\partial\alpha} + B(t) - R(t), \tag{2.6}$$

$$\frac{\partial B}{\partial t} = \frac{\partial B}{\partial\alpha} - b(t), \tag{2.7}$$

which we can be differentiated with respect α

$$\begin{aligned}
\frac{\partial^2 b}{\partial t\partial\alpha} &= \frac{\partial^2 b}{\partial\alpha^2} + \frac{\partial B}{\partial\alpha}, \\
\frac{\partial^2 B}{\partial t\partial\alpha} &= \frac{\partial^2 B}{\partial\alpha^2} - \frac{\partial b}{\partial\alpha},
\end{aligned} \tag{2.8}$$

and written as

$$\begin{aligned}
\frac{\partial^2 b}{\partial t\partial\alpha} &= \frac{\partial^2 b}{\partial\alpha^2} + R(t+\alpha-\beta-\gamma)\sin(\alpha-\beta-\gamma), \\
\frac{\partial^2 B}{\partial t\partial\alpha} &= \frac{\partial^2 B}{\partial\alpha^2} - R(t+\alpha-\beta-\gamma)\cos(\alpha-\beta-\gamma).
\end{aligned}$$

Since

$$\begin{aligned}
\frac{\partial^2 b}{\partial t\partial\alpha} &= \frac{\partial^2 b}{\partial\alpha\partial t} = \dot{R}(t+\alpha-\beta-\gamma)\cos(\alpha-\beta-\gamma), \\
\frac{\partial^2 B}{\partial t\partial\alpha} &= \frac{\partial^2 B}{\partial t\partial\alpha} = \dot{R}(t+\alpha-\beta-\gamma)\sin(\alpha-\beta-\gamma),
\end{aligned}$$

then we have

$$\begin{aligned}\frac{\partial^2 b}{\partial\alpha^2}-\frac{\partial^2 b}{\partial\alpha\partial t} &= -R(t+\alpha-\beta-\gamma)\sin(\alpha-\beta-\gamma),\\ \frac{\partial^2 B}{\partial\alpha^2}-\frac{\partial^2 B}{\partial\alpha\partial t} &= R(t+\alpha-\beta-\gamma)\cos(\alpha-\beta-\gamma).\end{aligned}$$

Let us diferentiate the equation (2.7) with respect t to get

$$\begin{aligned}\frac{\partial^2 B}{\partial t^2} &= \frac{\partial^2 B}{\partial\alpha\partial t}-\frac{\partial b}{\partial t},\\ \frac{\partial^2 B}{\partial t^2} &- \frac{\partial^2 B}{\partial\alpha\partial t}+B(t)=R(t)-R(t+\alpha-\beta-\gamma).\end{aligned}$$

Theorem 2.5. *The function* $u=\frac{1}{2}(b^2+B^2)$ *satisfy the following equation*

$$\frac{\partial^2 u}{\partial\alpha^2}-\frac{\partial^2 u}{\partial\alpha\partial t}=R(t)R(t+\alpha-\beta-\gamma)\cos(\alpha-\beta-\gamma). \tag{2.9}$$

Proof. Let us differentiate the expression $u=\frac{1}{2}(b^2+B^2)$

$$\begin{aligned}\frac{\partial u}{\partial\alpha} &= b(t)R(t+\alpha-\beta-\gamma)\cos(\alpha-\beta-\gamma)+\\ &+ B(t)R(t+\alpha-\beta-\gamma)\sin(\alpha-\beta-\gamma),\\ \frac{\partial^2 u}{\partial\alpha^2} &= R^2(t+\alpha-\beta-\gamma)+b(t)\left(\frac{\partial^2 b}{\partial t\partial\alpha}-R(t+\alpha-\beta-\gamma)\cos(\alpha-\beta-\gamma)\right)+\\ &+ B(t)\left(\frac{\partial^2 B}{\partial t\partial\alpha}-R(t+\alpha-\beta-\gamma)\sin(\alpha-\beta-\gamma)\right),\\ \frac{\partial^2 u}{\partial\alpha\partial t} &= \frac{\partial b}{\partial t}\frac{\partial b}{\partial\alpha}+b(t)\frac{\partial^2 b}{\partial\alpha\partial t}+\frac{\partial B}{\partial t}\frac{\partial B}{\partial\alpha}+B(t)\frac{\partial^2 B}{\partial\alpha\partial t}\end{aligned}$$

and let us consider the difference

$$\frac{\partial^2 u}{\partial\alpha^2}-\frac{\partial^2 u}{\partial\alpha\partial t} = R(t)R(t+\alpha-\beta-\gamma)\cos(\alpha-\beta-\gamma).$$

Hence we get (2.9). □

Theorem 2.6. *The function* $v=\sqrt{2u}$ *satisfy the folloving equation*

$$v\left(\frac{\partial^2 v}{\partial\alpha^2}-\frac{\partial^2 v}{\partial\alpha\partial t}\right)+\left(\frac{\partial v}{\partial\alpha}\right)^2-\frac{\partial v}{\partial t}\frac{\partial v}{\partial\alpha}=R(t)R(t+\alpha-\beta-\gamma)\cos(\alpha-\beta-\gamma). \tag{2.10}$$

Proof. Let us differentiate the function $v = \sqrt{2u}$

$$\begin{aligned}
\frac{\partial v}{\partial \alpha} &= \frac{1}{2\sqrt{2u}} 2\frac{\partial u}{\partial \alpha} = \frac{1}{\sqrt{2u}}\frac{\partial u}{\partial \alpha} = \frac{1}{v}\left(b(t)\frac{\partial b}{\partial \alpha} + B(t)\frac{\partial B}{\partial \alpha}\right) = \frac{1}{v}\frac{\partial u}{\partial \alpha}, \\
\frac{\partial v}{\partial t} &= \frac{1}{\sqrt{2u}}\frac{\partial u}{\partial t} = \frac{1}{v}\frac{\partial u}{\partial t}, \\
\frac{\partial^2 v}{\partial \alpha^2} &= \frac{1}{v^2}\left(v\frac{\partial^2 u}{\partial \alpha^2} - \frac{\partial v}{\partial \alpha}\frac{\partial u}{\partial \alpha}\right) = \frac{1}{v^2}\left(v\frac{\partial^2 u}{\partial \alpha^2} - \frac{1}{v}\left(\frac{\partial u}{\partial \alpha}\right)^2\right), \\
\frac{\partial^2 v}{\partial \alpha \partial t} &= \frac{1}{v^2}\left(v\frac{\partial^2 u}{\partial \alpha \partial t} - \frac{\partial v}{\partial t}\frac{\partial u}{\partial \alpha}\right) = \frac{1}{v^2}\left(v\frac{\partial^2 u}{\partial \alpha \partial t} - \frac{1}{v}\frac{\partial u}{\partial t}\frac{\partial u}{\partial \alpha}\right)
\end{aligned}$$

and let us consider the expression

$$\frac{\partial^2 v}{\partial \alpha^2} - \frac{\partial^2 v}{\partial \alpha \partial t} = \frac{1}{v}\left(\frac{\partial^2 u}{\partial \alpha^2} - \frac{\partial^2 u}{\partial \alpha \partial t} - \frac{1}{v^2}\left(\frac{\partial u}{\partial \alpha}\right)^2 + \frac{1}{v^2}\frac{\partial u}{\partial t}\frac{\partial u}{\partial \alpha}\right).$$

Using (2.9) we have

$$v\left(\frac{\partial^2 v}{\partial \alpha^2} - \frac{\partial^2 v}{\partial \alpha \partial t}\right) = R(t)R(t+\alpha-\beta-\gamma)\cos(\alpha-\beta-\gamma) - \left(\frac{\partial v}{\partial \alpha}\right)^2 + \frac{\partial v}{\partial t}\frac{\partial v}{\partial \alpha},$$

and we have

$$v\left(\frac{\partial^2 v}{\partial \alpha^2} - \frac{\partial^2 v}{\partial \alpha \partial t}\right) + \left(\frac{\partial v}{\partial \alpha}\right)^2 - \frac{\partial v}{\partial t}\frac{\partial v}{\partial \alpha} = R(t)R(t+\alpha-\beta-\gamma)\cos(\alpha-\beta-\gamma),$$

where $v = |q(t)| = \sqrt{b^2(t) + B^2(t)}$. $\square$

Let us define the function $F(\alpha)$ by the formula

$$F(\alpha) = \int_0^{2\pi} R(t)p(t+\alpha-\beta-\gamma)dt. \tag{2.11}$$

Then

$$\dot{F}(\alpha) = \int_0^{2\pi} R(t)\dot{p}(t+\alpha-\beta-\gamma)dt, \tag{2.12}$$

$$\ddot{F}(\alpha) = \int_0^{2\pi} R(t)\ddot{p}(t+\alpha-\beta-\gamma)dt. \tag{2.13}$$

From the formula (2.9) we have

$$(F(\alpha) + \ddot{F}(\alpha))\cos(\alpha-\beta-\gamma) = \frac{1}{2}\frac{d^2}{d\alpha^2}\int_0^{2\pi} |q(\alpha,t)|^2 dt.$$

If we put

$$W(\alpha) = \frac{1}{2}\frac{d^2}{d\alpha^2}\int_0^{2\pi} |q(\alpha, t)|^2 dt,$$

then the function $F(\alpha)$ satisfy the equation

$$(F(\alpha) + \ddot{F}(\alpha))\cos(\alpha - \beta - \gamma) = \ddot{W}(\alpha). \tag{2.14}$$

If

$$\alpha = \frac{\pi}{2} + \beta + \gamma,$$

then $\cos(\alpha - \beta - \gamma) = 0$. Hence $\ddot{W}(\frac{\pi}{2} + \beta + \gamma) = 0$.

References

[1] K. Benko, W. Cieślak, S. Góźdź, and W. Mozgawa, *On isoptic curves,* An. Ştiinţ. Univ. Al. I. Cuza Iaşi Secţ. I a Mat. **36**, no. 1 (1990), 47–54.

[2] J. Berg, A. Yezzi, and A. Tannenbaum, *Curve evolution models for real-time identification with application to plasma etching*, IEEE Trans. Autom. Control **44**, no. 1 (1999), 99–102.

[3] W. Cieślak, A. Miernowski, and W. Mozgawa, *Isoptics of a closed strictly convex curve,* Global differential geometry and global analysis (Berlin 1990), 28–35, Lecture Notes in Math. **1481**, Springer, Berlin 1991.

[4] W. Cieślak, A. Miernowski, and W. Mozgawa, *Isoptics of a closed strictly convex curve II,* Rend. Sem. Mat. Univ. Padova **96** (1996), 37–49.

[5] M. Gage, *On an area-preserving evolution equation for plane curves,* Contemp. Math. **51** (1986), 51–62.

[6] P. J. Giblin and B. B. Kimia, *Local forms and transitions of the medial axis*, Springer. Computational Imaging and Vision **37** (2008), 37–68.

[7] B. B. Kimia, A. Tannenbaum, and S. W. Zucker, *On the evolution of curves via a function of curvature. I: The classical case,* J. Math. Anal. Appl. **163**, no. 2, (1992), 438–458.

[8] F. F. Leymarie and B. B. Kimia, *From the infinitely large to the infinitely small*, Springer. Computational Imaging and Vision **37** (2008), 327–351.

[9] W. Mozgawa and M. Skrzypiec, *Crofton formulas and convexity condition for secantoptics*, in print Bull. Belg. Math. Soc. – Simon Stevin.

[10] W. Mozgawa and M. Skrzypiec, *Some properties of secantoptics of ovals,* Beiträge Algebra Geom. **53**, no. 1 (2012), 261–272.

[11] W. Mozgawa and M. Skrzypiec, *Integral formula for secantoptics and its application*, Ann. Mariae Curie-Skłodowska Univ., Sect. A **66**, no. 1 (2012), 49–62.

[12] P. J. Olver, G. Sapiro, and A. Tannenbaum, *Invariant geometric evolutions of surfaces and volumetric smoothing*, SIAM J. Appl. Math. **57**, no. 1, (1997), 176–194.

[13] L. Santalo, *Integral geometry and geometric probability,* Encyclopedia of Mathematics and its Applications, Reading, Mass., 1976.

[14] M. Skrzypiec, *A note on secantoptics,* Beiträge Algebra Geom. **49**, no. 1 (2008), 205–215.

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RÓWNANIA RÓŻNICZKOWE DLA SEKANTOOPTYK

S t r e s z c z e n i e

Zbiór punktów przecięcia odpowiedniej pary siecznych danego owalu C, tworzy krzywą, którą nazwaliśmy sekantooptyką $C_{\alpha,\beta,\gamma}$ owalu C. W pracach [14], [9], [10] i [11] dowiedliśmy wielu własności sekantooptyk i znaleźliśmy zależności pomiędzy pojęciami sekantooptyk, ewolutoid, jeży oraz izooptyk pary krzywych. W tej pracy prezentujemy pewne równania różniczkowe związane z sekantooptykami.

Słowa kluczowe: izooptyka, sekantooptyka, obwiednia

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Arezki Touzaline

VARIATIONAL ANALYSIS OF A UNILATERAL CONTACT PROBLEM WITH ADHESION AND SLIP-DEPENDENT FRICTION

Summary

The aim of this paper is to study a quasistatic contact between an elastic body and a foundation. The constitutive law is nonlinear and the contact is modelled with unilaeral constraint and normal compliance, associated with a slip-dependent version of Coulomb's law of dry friction. The adhesion between contact surfaces is taken into account and is modelled with a surface variable, the bonding field, whose evolution is described by a first-order differential equation. We establish a variational formulation of the mechanical problem and prove an existence and uniqueness result. The technique of the proof is based on arguments of time-dependent variational inequalities, differential equations and Banach fixed-point theorem.

Keywords and phrases: elastic, normal compliance, adhesion, friction, unilateral constraint

1. Introduction

Contact problems involving deformable bodies are quite frequent in industry as well as in daily life and play an important role in structural and mechanical systems. Contact processes involve complicated surface phenomena, and are modelled with highly nonlinear initial boundary value problems. Taking into account various contact conditions associated with more and more complex behavior laws lead to the introduction of new and non standard models, expressed by the aid of evolution variational inequalities. An early attempt to study contact problems within the framework of variational inequalities was made in [11]. The mathematical, mechanical and numerical state of the art can be found in [27] where we find detailed mathematical and numerical studies of the adhesive contact problems. Unilateral contact problems involving Signorini's condition with or without adhesion were studied by several authors, see for instance the papers [1–3, 6–10, 12, 18, 19, 22, 23, 27, 31–34].

In this paper, we study a mathematical model which describes a frictional contact problem with adhesion be.tween a nonlinear elastic body and a deformable foundation. Following [18, 33] the contact is modelled with unilateral constraint and normal compliance, associated to the Coulomb's law where the coefficient of friction depends on the slip displacement. This assumption is used by geological researchers in the study of the motion of tectonic plates; see [17, 21, 25] and the references therein. Moreover the adhesion between the contact surfaces is taken into account. Recall that this model without adhesion was studied recently in [2] for elastic materials. However the models for dynamic or quasistatic processes of adhesive contact between a deformable body and a foundation have been studied in [3–5, 7, 8, 13, 16, 19, 20, 22, 24, 26–33] and the references therein. Here as in [14, 15] we use the bonding field as an additional state variable β, defined on the contact surface of the boundary. The variable is restricted to values $0 \leq \beta \leq 1$; when $\beta = 0$ all the bonds are severed and there are no active bonds, when $\beta = 1$ all the bonds are active; when $0 < \beta < 1$ it measures the fraction of active bonds and partial adhesion takes place.

In this work we extend the result established in [29] to the unilateral and adhesive contact problem with a normal compliance condition, associated with a slip-dependent version of Coulomb's law of dry friction. We establish a variational formulation of the mechanical problem for which we prove the existence of a unique weak solution.

The paper is structured as follows. In section 2 we present some notations and give the variational formulation. In section 3 we state and prove our main existence and uniqueness result, Theorem 3.1.

2. Problem statement and variational formulation

In this section we describe the model of the process, list the assumptions on the data and derive the variational formulation of the mechnical problem. The physical setting is as follows. A nonlinear elastic body occupies a domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with a Lipschiz boundary Γ that is divised into three measurable and disjoint parts $\Gamma_1, \Gamma_2, \Gamma_3$ such that $meas\,(\Gamma_1) > 0$. The body is acted upon by a volume force of density f_1 on Ω and a surface traction of density f_2 on Γ_2. The body is in adhesive contact with a fondation over Γ_3 following a slip-dependent version of Coulomb's law of dry friction.

Thus, the classical formulation of the elastic contact problem with adhesion and slip-dependent friction is the following.

Problem P_1. Find a displacement $u : \Omega \times [0, T] \to \mathbb{R}^d$ and a bonding field $\beta : \Gamma_3 \times [0, T] \to \mathbb{R}$ such that

$$div\sigma(u) = -f_1 \text{ in } \Omega \times (0,T), \tag{2.1}$$

$$\sigma(u) = F\varepsilon(u) \quad \text{in } \Omega \times (0,T), \tag{2.2}$$

$$u = 0 \quad \text{on } \Gamma_1 \times (0,T), \tag{2.3}$$

$$\sigma\nu = f_2 \quad \text{on } \Gamma_2 \times (0,T), \tag{2.4}$$

$$\left.\begin{array}{l} u_\nu \le g,\ \sigma_\nu + p(u_\nu) - c_\nu\beta^2 R_\nu(u_\nu) \le 0 \\ \\ \left(\sigma_\nu + p(u_\nu) - c_\nu\beta^2 R_\nu(u_\nu)\right)(u_\nu - g) = 0 \end{array}\right\} \text{ on } \Gamma_3 \times (0,T), \tag{2.5}$$

$$\left.\begin{array}{l} \left|\sigma_\tau + c_\tau\beta^2 R_\tau(u_\tau)\right| \le \mu(|u_\tau|)\, p(u_\nu) \\ \left|\sigma_\tau + c_\tau\beta^2 R_\tau(u_\tau)\right| < \mu(|u_\tau|)\, p(u_\nu) \Rightarrow u_\tau = 0 \\ \left|\sigma_\tau + c_\tau\beta^2 R_\tau(u_\tau)\right| = \mu(|u_\tau|)\, p(u_\nu) \Rightarrow \exists\lambda \ge 0 \text{ such that} \\ \sigma_\tau + c_\tau\beta^2 R_\tau(u_\tau) = -\lambda u_\tau \end{array}\right\} \text{on } \Gamma_3 \times (0,T), \tag{2.6}$$

$$\dot\beta = -\left[\beta\left(c_\nu(R_\nu(u_\nu))^2 + c_\tau\left|R_\tau(u_\tau)\right|^2\right) - \varepsilon_a\right]_+ \text{ on } \Gamma_3 \times (0,T), \tag{2.7}$$

$$\beta(0) = \beta_0 \text{ on } \Gamma_3. \tag{2.8}$$

Equation (2.1) represents the equilibrium equation. Equation (2.2) represents the elastic constitutive law of the material in which F is a given function and $\varepsilon(u)$ denotes the strain tensor; (2.3) and (2.4) are the displacement and traction boundary conditions, respectively, in which ν denotes the unit outward normal vector on Γ and $\sigma\nu$ represents the Cauchy stress vector. The condition (2.5) represents the unilateral contact with adhesion in which p and $-c_\nu\beta^2 R_\nu(u_\nu)$ are the normal contact functions. Here c_ν is a given adhesion coefficient and R_ν is a truncation operator defined by

$$R_\nu(s) = \left\{\begin{array}{l} L \ \text{ if } s < -L \\ -s \text{ if } -L \le s \le 0 \\ 0 \text{ if } s > 0 \end{array}\right. .$$

Here $L > 0$ is the characteristic length of the bond, beyond which the latter has no additional traction (see [27]) and p is a normal compliance function which satisfies the assumption beow (2.16). We denote by the positive constant g the maximum value of the penetration. When $u_\nu < 0$ i.e. when there is separation between the body and the foundation then the condition (2.5) combined with hypotheses (2.16) on the function p shows that $\sigma_\nu = c_\nu\beta^2 R_\nu(u_\nu)$ and by assumption (2.17) below, it does not exeed the value $L(1+g)$. When $g > 0$, the body may interpenetrate into the foundation, but the penetration is limited that is $u_\nu \le g$. In this case of penetration (i.e. $u_\nu \ge 0$), when $0 \le u_\nu < g$ then $-\sigma_\nu = p(u_\nu)$ which means that the reaction of the foundation is uniquely determined by the normal displacement and $\sigma_\nu \le 0$. Since p is an increasing function then the reaction of the foundation is increasing with the penetration and when $u_\nu = g$, then $-\sigma_\nu \ge p(g)$ and σ_ν is not uniquely determined. When $g > 0$ and $p = 0$, condition (2.5) becomes the Signorini contact condition with adhesion with a gap function,

$$u_\nu \le g,\ \sigma_\nu - c_\nu \beta^2 R_\nu(u_\nu) \le 0,\ (\sigma_\nu - c_\nu \beta^2 R_\nu(u_\nu))(u_\nu - g) = 0.$$

When $g = 0$, the condition (2.5) combined with assumption (2.16) becomes the Signorini contact condition with adhesion with a zero gap function, given by

$$u_\nu \le 0,\ \sigma_\nu - c_\nu \beta^2 R_\nu(u_\nu) \le 0,\ (\sigma_\nu - c_\nu \beta^2 R_\nu(u_\nu))u_\nu = 0.$$

This contact condition was used in [7, 25, 26, 29, 30]. The condition (2.6) represents the slip-dependent version of Coulomb's friction law with adhesion in which $c_\tau \beta^2 R_\tau(u_\tau)$ is an adhesive where c_τ is a coefficient of adhesion and R_τ is a truncation operator defined by

$$R_\tau(v) = \begin{cases} v \ \text{ if } |v| \le L \\ L\dfrac{v}{|v|} \text{ if } |v| > L \end{cases},$$

where $L > 0$ is the characteristic length of the bonds. Equation (2.7) represents the ordinary differential equation which describes the evolution of the bonding field, where ε_a is an adhesion coefficient and $\beta_+ = \max(0, \beta)$. Since $\dot{\beta} \le 0$ on $\Gamma_3 \times (0, T)$, once debonding occurs bonding cannot be reestablished and, indeed, the adhesive process is irreversible. Also from [20] it must be pointed out clearly that condition (2.7) does not allow for complete debonding in finite time. Finally, (2.8) is the initial condition, in which β_0 denotes the initial bonding field. In (2.7) a dot above a variable represents its derivative with respect to time. We denote by S_d the space of second order symmetric tensors on $\mathbb{R}^d$ $(d = 2, 3)$ and $|.|$ represents the Euclidean norm on $\mathbb{R}^d$ and S_d. Thus, for every $u, v \in \mathbb{R}^d$, $u.v = u_i v_i$, $|v| = (v.v)^{\frac{1}{2}}$, and for every $\sigma, \tau \in S_d$, $\sigma.\tau = \sigma_{ij}\tau_{ij}$, $|\tau| = (\tau.\tau)^{\frac{1}{2}}$. Here and below, the indices i and j run between 1 and d and the summation convention over repeated indices is adopted. Now, to proceed with the variational formulation, we need the following function spaces:

$$H = \left(L^2(\Omega)\right)^d,\ H_1 = \left(H^1(\Omega)\right)^d,\ Q = \left\{\tau = (\tau_{ij});\ \tau_{ij} = \tau_{ji} \in L^2(\Omega)\right\},$$
$$Q_1 = \{\tau \in Q;\ div\tau \in H\}.$$

Note that H and Q are real Hilbert spaces endowed with the respective canonical inner products

$$(u, v)_H = \int_\Omega u_i v_i dx, \quad (\sigma, \tau)_Q = \int_\Omega \sigma_{ij}\tau_{ij} dx.$$

The strain tensor is

$$\varepsilon(u) = (\varepsilon_{ij}(u)) = \frac{1}{2}(u_{i,j} + u_{j,i})$$

and $div\sigma = (\sigma_{ij,j})$ is the divergence of σ. For every $v \in H_1$ we denote by v_ν and v_τ the normal and tangential components of v on the boundary Γ given by

$$v_\nu = v.\nu, \quad v_\tau = v - v_\nu \nu.$$

We also denote by σ_ν and σ_τ the normal and the tangential traces of a function $\sigma \in Q_1$, and when σ is a regular function then

$$\sigma_\nu = (\sigma\nu).\nu, \quad \sigma_\tau = \sigma\nu - \sigma_\nu \nu,$$

and the following Green's formula holds:

$$(\sigma, \varepsilon(v))_Q + (div\sigma, v)_H = \int_\Gamma \sigma\nu.vda \ \ \forall v \in H_1,$$

where da is the surface measure element. Now, let V be the closed subspace of H_1 defined by

$$V = \{v \in H_1 : v = 0 \text{ on } \Gamma_1\},$$

and let the convex subset of admissible displacements given by

$$K = \{v \in V : v_\nu \leq g \ a.e \text{ on } \Gamma_3\}.$$

Since $meas(\Gamma_1) > 0$, the following Korn's inequality holds [11],

$$\|\varepsilon(v)\|_Q \geq c_\Omega \|v\|_{H_1} \quad \forall v \in V, \tag{2.9}$$

where $c_\Omega > 0$ is a constant which depends only on Ω and Γ_1. We equip V with the inner product

$$(u, v)_V = (\varepsilon(u), \varepsilon(v))_Q$$

and $\|.\|_V$ is the associated norm. It follows from Korn's inequality (2.9) that the norms $\|.\|_{H_1}$ and $\|.\|_V$ are equivalent on V. Then $(V, \|.\|_V)$ is a real Hilbert space. Moreover by Sobolev's trace theorem, there exists $d_\Omega > 0$ which only depends on the domain Ω, Γ_1 and Γ_3 such that

$$\|v\|_{(L^2(\Gamma_3))^d} \leq d_\Omega \|v\|_V \quad \forall v \in V. \tag{2.10}$$

For $p \in [1, \infty]$, we use the standard norm of $L^p(0, T; V)$. We also use the Sobolev space $W^{1,\infty}(0, T; V)$ equipped with the norm

$$\|v\|_{W^{1,\infty}(0,T;V)} = \|v\|_{L^\infty(0,T;V)} + \|\dot{v}\|_{L^\infty(0,T;V)}.$$

For every real Banach space $(X, \|.\|_X)$ and $T > 0$ we use the notation $C([0, T]; X)$ for the space of continuous functions from $[0, T]$ to X; recall that $C([0, T]; X)$ is a real Banach space with the norm

$$\|x\|_{C([0,T];X)} = \max_{t \in [0,T]} \|x(t)\|_X.$$

We assume that the body forces and surface tractions have the regularity

$$f_1 \in W^{1,\infty}(0, T; H), \quad f_2 \in W^{1,\infty}\left(0, T; (L^2(\Gamma_2))^d\right). \tag{2.11}$$

It follows from (2.11) and Riesz's representation theorem that there exists a function $f : [0, T] \to V$ such that

$$(f(t), v)_V = \int_\Omega f_1(t).vdx + \int_{\Gamma_2} f_2(t).vda \ \ \forall v \in V, \, t \in [0, T], \tag{2.12}$$

and note that (2.11) and (2.12) imply that

$$f \in W^{1,\infty}(0, T; V).$$

In the study of the mechanical problem P_1 we suppose that the nonlinear elasticity operator $F : \Omega \times S_d \to S_d$ satisfies

$$(2.13)\qquad \left.\begin{array}{l} (a) \text{ there exists } M > 0 \text{ such that} \\ \quad |F(x,\varepsilon_1) - F(x,\varepsilon_2)| \leq M\,|\varepsilon_1 - \varepsilon_2| \quad \forall\, \varepsilon_1, \varepsilon_2 \in S_d, \\ \quad a.e.\ x \in \Omega; \\ \\ (b) \text{ there exists } m > 0 \text{ such that} \\ \quad (F(x,\varepsilon_1) - F(x,\varepsilon_2)) \,.\, (\varepsilon_1 - \varepsilon_2) \geq m\,|\varepsilon_1 - \varepsilon_2|^2, \\ \quad \forall \varepsilon_1, \varepsilon_2 \in S_d,\ a.e.\ x \in \Omega; \\ \\ (c) \text{ the mapping } x \to F(x,\varepsilon) \text{ is Lebesgue measurable on } \Omega, \\ \quad \text{for any } \varepsilon \in S_d; \\ \\ (d)\ F(x,0) = 0 \text{ for } a.e.\ x \in \Omega. \end{array}\right\}$$

The adhesion coefficients satisfy

$$(2.14)\qquad c_\nu, c_\tau \in L^\infty(\Gamma_3),\ \varepsilon_a \in L^2(\Gamma_3) \text{ and } c_\nu, c_\tau, \varepsilon_a \geq 0 \quad a.e. \text{ on } \Gamma_3,$$

and the coefficient of friction μ is assumed to satisfy

$$(2.15)\qquad \left.\begin{array}{l} (a)\ \mu : \Gamma_3 \times \mathbb{R}_+ \to \mathbb{R}_+; \\ \\ (b) \text{ there exists } L_\mu > 0 \text{ such that} \\ \quad |\mu(x,r_1) - \mu(x,r_2)| \leq L_\mu\,|r_1 - r_2| \quad \forall r_1, r_2 \in \mathbb{R}_+,\ a.e.\ x \in \Gamma_3; \\ \\ (c) \text{ there exists } \mu_0 > 0 \text{ such that} \\ \quad \mu(x,r) \leq \mu_0 \ \ \forall r \in \mathbb{R}_+,\ a.e.\ x \in \Gamma_3; \\ \\ (d) \text{ the function } x \to \mu(x,r) \text{ is Lebesgue} \\ \quad \text{measurable on } \Gamma_3,\ \forall r \in \mathbb{R}_+. \end{array}\right\}$$

Also we define respectively the functionals

$$j_{ad} : L^2(\Gamma_3) \times V \times V \to \mathbb{R} \quad \text{and} \quad j_{fr} : V \times V \to \mathbb{R}_+$$

by

$$j_{ad}(\beta,u,v) = \int_{\Gamma_3} \left[(p(u_\nu) - c_\nu \beta^2 R_\nu(u_\nu)) v_\nu + c_\tau \beta^2 R_\tau(u_\tau) \,.\, v_\tau \right] da,$$

$$\forall (\beta,u,v) \in L^2(\Gamma_3) \times V \times V,$$

$$j_{fr}(u,v) = \int_{\Gamma_3} \mu(|u_\tau|)\, p(u_\nu)\, |v_\tau|\, da \quad \forall (u,v) \in V \times V,$$

where the normal compliance function $p : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ satisfies

$$(2.16)\quad \begin{cases} (a)\ \text{There exists } L_1 > 0 \text{ such that} \\ \quad |p(x, r_1) - p(x, r_2)| \le L_1 |r_1 - r_2| \\ \quad \forall r_1,\ r_2 \in \mathbb{R},\ a.e.\ x \in \Gamma_3; \\ \\ (b)\ (p(x, r_1) - p(x, r_2))(r_1 - r_2) \ge 0 \\ \quad \forall r_1,\ r_2 \in \mathbb{R},\ a.e.\ x \in \Gamma_3; \\ \\ (c)\ \text{there exists } L_2 > 0 \text{ such that} \\ \quad p(x, r) \le L_2 \quad \forall r \in \mathbb{R},\ a.e.\ x \in \Gamma_3; \\ \\ (d)\ \text{the mapping } x \to p_\nu(x, r) \text{ is measurable on } \Gamma_3, \text{ for any } r \in \mathbb{R}; \\ \\ (e)\ p(x, r) = 0\ \forall r \le 0,\ a.e.\ x \in \Gamma_3. \end{cases}$$

Finally, we assume that the initial data satisfies

$$(2.17)\qquad \beta_0 \in L^2(\Gamma_3);\ 0 \le \beta_0 \le 1\ a.e.\ \text{on } \Gamma_3$$

and next, we need to introduce the following set for the bonding field,

$$\mathcal{B} = \left\{\theta : [0, T] \to L^2(\Gamma_3); 0 \le \theta(t) \le 1,\ \forall t \in [0, T],\ a.e.\ \text{on } \Gamma_3\right\}.$$

Now, by a standard procedure based on the Green formula, we obtain the following variational formulation of the mechanical problem P_1.

Problem P_2. Find a displacement field $u : [0, T] \to V$ and a bonding field $\beta : [0, T] \to L^2(\Gamma_3)$ such that

(2.18)

$$u(t) \in K,\ (F\varepsilon(u(t)), \varepsilon(v) - \varepsilon(u(t)))_Q + j_{ad}(\beta(t), u(t), v - u(t))$$

$$+ j_{fr}(u(t), v) - j_{fr}(u(t), u(t)) \ge (f(t), v - u(t))_V \quad \forall\, v \in K,\ t \in [0, T],$$

$$(2.19)\quad \dot{\beta}(t) = -\left[\beta(t)\left(c_\nu (R_\nu(u_\nu(t)))^2 + c_\tau |R_\tau(u_\tau(t))|^2\right) - \varepsilon_a\right]_+ \quad a.e.\ t \in (0, T),$$

$$(2.20)\qquad \beta(0) = \beta_0.$$

3. Existence of solution for Problem P_2

Our main existence and uniqueness result concerning Problem P_2 which we establish in this section, is the following.

Theorem 3.1. *Let* (2.11), (2.13), (2.14), (2.15), (2.16) *and* (2.17) *hold. Then Problem* P_2 *has a unique solution, which satisfies*

$$(3.1)\qquad u \in W^{1,\infty}(0, T; V) \ \cap C([0, T]; K) \ \ and$$

$$\beta \in W^{1,\infty}\left(0,T;L^2\left(\Gamma_3\right)\right) \cap \mathcal{B}, \tag{3.2}$$

if

$$\left(L_2 L_\mu + \mu_0 L_1\right) < m/d_\Omega^2.$$

We assume in the following that conditions of Theorem 3.1 hold; below, c denotes a generic positive constant which does not depend on t nor on the rest of the input data, and whose value may change from place to place. The proof of the theorem is carried out in several steps. In the first step, let $k > 0$ and consider the closed subset X of $C\left([0,T];L^2\left(\Gamma_3\right)\right)$ defined as

$$X = \left\{\theta \in C\left([0,T];L^2\left(\Gamma_3\right)\right) \cap B,\ \theta\left(0\right) = \beta_0\right\},$$

where the Banach space $C\left([0,T];L^2\left(\Gamma_3\right)\right)$ is endowed with the norm

$$\|\theta\|_k = \max_{t\in[0,T]}\left[\exp\left(-kt\right)\|\theta\left(t\right)\|_{L^2(\Gamma_3)}\right] \text{ for all } \theta \in C\left([0,T];L^2\left(\Gamma_3\right)\right).$$

Next for a given $\beta \in X$, we consider the following variational problem.

Problem $P_{1\beta}$. Find $u_\beta : [0,T] \to V$ such that

$$\begin{aligned} &u_\beta\left(t\right) \in K,\ \left(F\varepsilon\left(u_\beta\left(t\right)\right),\varepsilon\left(v - u_\beta\left(t\right)\right)\right)_Q + j_{ad}\left(\beta\left(t\right),u_\beta\left(t\right),v - u_\beta\left(t\right)\right) \\ &+j_{fr}\left(u_\beta\left(t\right),v\right) - j_{fr}\left(u_\beta\left(t\right),u_\beta\left(t\right)\right) \geq \left(f\left(t\right),v - u_\beta\left(t\right)\right)_V \quad \forall v \in K,\ t \in [0,T]. \end{aligned} \tag{3.3}$$

We have the following result.

Proposition 3.2. *Problem* $P_{1\beta}$ *has a unique solution*

$$u_\beta \in C\left([0,T];K\right), \tag{3.4}$$

if

$$\left(L_2 L_\mu + \mu_0 L_1\right) < m/d_\Omega^2.$$

We shall establish the proof of Proposition 3.2 in several steps. Indeed, at first for each $t \in [0,T]$ and a given $\eta \in K$, we consider the following auxiliary problem.

Problem $P_{\beta\eta}$. Find $u_{\beta\eta}\left(t\right) \in K$ such that

$$\begin{aligned} &\left(F\varepsilon\left(u_{\beta\eta}\left(t\right)\right),\varepsilon\left(v - u_{\beta\eta}\left(t\right)\right)\right)_Q + j_{ad}\left(\beta\left(t\right),u_{\beta\eta}\left(t\right),v - u_{\beta\eta}\left(t\right)\right) + j\left(\eta,v\right) \\ &-j_{fr}\left(\eta,u_{\beta\eta}\left(t\right)\right) \geq \left(f\left(t\right),v - u_{\beta\eta}\left(t\right)\right)_V \quad \forall v \in K. \end{aligned} \tag{3.5}$$

We have the lemma below.

Lemma 3.3. *Problem* $P_{\beta\eta}$ *has a unique solution.*

Proof. Let the operator $A_t : V \to V$ defined by

$$\left(A_t u,v\right)_V = \left(F\varepsilon\left(u\right),\varepsilon\left(v\right)\right)_Q + j_{ad}\left(\beta\left(t\right),u,v\right),\ \forall u,v \in V.$$

We use (2.10), (2.13a), (2.13b), (2.16b) and (2.16c) to show that the operator A_t is strongly monotone and Lipschitz continuous; the functional $j(\eta,.) : K \to \mathbb{R}_+$ is a continuous seminorm; then by a standard existence and uniqueness result for elliptic quasivariational inequalities (see [29]), it follows that there exists a unique element $u_{\beta\eta}(t) \in K$ which satisfies the inequality (3.5) since K is a non-empty, closed convex subset of V. □

Now, in the second step, for a fixed $t \in [0,T]$ we use Lemma 3.3 to consider the map $T_t : K \to K$ defined as

$$T_t(\eta) = u_{\beta\eta}(t).$$

We have the following lemma.

Lemma 3.4. *The map T_t has a unique fixed point η^* and $u_{\beta\eta^*}(t)$ is a unique solution of the inequality* (3.3).

Proof. Let $\eta_1, \eta_2 \in K$. In inequality (3.5) satisfied by $u_{\eta_1}(t)$ take $v = u_{\eta_2}(t)$ and also in the same inequality satisfied by $u_{\eta_2}(t)$ take $v = u_{\eta_1}(t)$. Using (2.10), (2.13)(c) and (2.16) , it follows after adding the resulting inequalities that

$$\|T_t(\eta_1) - T_t(\eta_2)\|_V \leq \frac{(L_2 L_\mu + \mu_0 L_1)\, d_\Omega^2}{m} \|\eta_1 - \eta_2\|_V.$$

Then for $(L_2 L_\mu + \mu_0 L_1) < m/d_\Omega^2$, the map T_t is a contraction; so it has a unique fixed point η^* and $u_{\beta\eta^*}(t)$ is a unique solution of the inequality (3.3). Next, denote $u_{\beta\eta^*}(t) = u_\beta(t)$ for each $t \in [0,T]$. As in [28], to show that $u_\beta \in C([0,T];K)$, it suffices to see from (3.3) that under the condition $(L_2 L_\mu + \mu_0 L_1) < m/d_\Omega^2$, we have

$$\begin{aligned} &\|u_\beta(t_1) - u_\beta(t_2)\|_V \leq \\ &c\Big(\|f(t_1) - f(t_2)\|_V + \|\beta(t_1) - \beta(t_2)\|_{L^2(\Gamma_3)}\Big) \quad \forall t_1, t_2 \in [0,T]. \end{aligned} \tag{3.6}$$

Therefore, using the regularity

$$f \in C([0,T];V) \quad \text{and} \quad \beta \in C\big([0,T];\ L^2(\Gamma_3)\big),$$

we immediately obtain (3.4). □

In the second step, we use Lemma 3.4 to consider the following initial value problem.

Problem $P_{2\beta}$. Find $\chi_\beta : [0,T] \to L^2(\Gamma_3)$ such that

$$\dot{\chi}_\beta(t) = -\Big[\chi_\beta(t)\Big(c_\nu (R_\nu(u_{\beta\nu}(t)))^2 + c_\tau |R_\tau(u_{\beta\tau}(t))|^2\Big) - \varepsilon_a\Big]_+ \quad a.e.\ t \in (0,T), \tag{3.7}$$

$$\chi_\beta(0) = \beta_0. \tag{3.8}$$

We obtain the following result.

Lemma 3.5. *Problem $P_{2\beta}$ has a unique solution χ_β which satisfies*

$$\chi_\beta \in W^{1,\infty}\big(0,T;L^2(\Gamma_3)\big) \cap \mathcal{B}.$$

Proof. Consider the mapping

$$F_\beta(t,\theta) : [0,T] \times L^2(\Gamma_3) \to L^2(\Gamma_3)$$

defined by

$$F_\beta(t,\theta) = -\left[\theta\left(c_\nu (R_\nu(u_{\beta\nu}(t)))^2 + c_\tau |R_\tau(u_{\beta\tau}(t))|^2\right) - \varepsilon_a\right]_+ .$$

It follows from the properties of the truncation operators R_ν and R_τ, that F_β is Lipschitz continuous with respect to the second argument, uniformly in time. Moreover, for any $\theta \in L^2(\Gamma_3)$, the mapping $t \to F_\beta(t,\theta)$ belongs to $L^\infty\left(0,T;L^2(\Gamma_3)\right)$. Then, from a version of Cauchy-Lipschitz theorem, we deduce the existence of a unique fonction $\chi_\beta \in W^{1,\infty}\left(0,T;L^2(\Gamma_3)\right)$, which satisfies (3.7), (3.8). The regularity $\chi_\beta \in \mathcal{B}$, follows from (3.7), (3.8) and (2.20), (see [27]). Therefore, from Lemma 3.5, we deduce that for all $\beta \in X$, the solution χ_β of Problem $P_{2\beta}$ belongs to X. In the third step we use this lemma to define the map $\Lambda : X \to X$ by

$$\Lambda\beta = \chi_\beta.$$

We have the following lemma.

Lemma 3.6. *The map Λ has a unique fixed point β^*.*

Proof. Let u_β the solution of Problem $P_{1\beta}$. We have

$$\Lambda\beta(t) = \beta_0 - \int_0^t \left[\chi_\beta(s)\left(c_\nu (R_\nu(u_{\beta\nu}(s)))^2 + c_\tau |R_\tau(u_{\beta\tau}(s))|^2\right) - \varepsilon_a\right]_+ ds.$$

Then for β_1, $\beta_2 \in X$, by (2.19) (a) and the properties of R_ν and R_τ (see [27]), we get

$$|\chi_{\beta_1}(t) - \chi_{\beta_2}(t)| \leq$$

$$c\int_0^t (|\chi_{\beta_1}(s) - \chi_{\beta_2}(s)| + |u_{\beta_1\tau}(s) - u_{\beta_2\tau}(s)|)ds.$$

Applying Gronwall's inequality and using (2.10) yields

$$\|\chi_{\beta_1}(t) - \chi_{\beta_2}(t)\|_{L^2(\Gamma_3)} \leq c\int_0^t \|u_{\beta_1}(s) - u_{\beta_2}(s)\|_V \, ds.$$

Now let $t \in [0,T]$. Then, using (3.3), (2.13), (2.16) and

$$(L_2 L_\mu + \mu_0 L_1) < m/d_\Omega^2,$$

as in [31], it follows that

$$\|u_{\beta_1}(t) - u_{\beta_2}(t)\|_V \leq c\,\|\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma_3)}$$

from which we deduce the inequality

$$\|\Lambda\beta_1(t) - \Lambda\beta_2(t)\|_{L^2(\Gamma_3)} \leq c\int_0^t \|\beta_1(s) - \beta_2(s)\|_{L^2(\Gamma_3)} \, ds \quad \forall t \in [0,T]$$

and therefore, we obtain

$$\|\Lambda\beta_1 - \Lambda\beta_2\|_k \leq \frac{c}{k}\|\beta_1 - \beta_2\|_k\,, \ \forall \beta_1, \beta_2 \in X.$$

Thus, this inequality implies that for k sufficiently large, Λ is a contraction. Then, it has a unique fixed point β^* which satisfies (3.7) and (3.8). On the other hand from (3.6) and (2.11) we deduce that $u_{\beta^*} \in W^{1,\infty}(0,T;V)$. $\square$

Proof of Theorem 3.1. Let $\beta = \beta^*$ and u_{β^*} the solution to Problem $P_{1\beta}$. From (3.3), (3.7) and (3.8) we conclude that (u_{β^*}, β^*) is a solution of Problem P_2. To prove the uniqueness of solution, assume that (u, β) is a solution of Problem P_2 which satisfies (2.18), (2.19) and (2.20). It follows from (2.18) that u is a solution of Problem $P_{1\beta}$ and by Proposition 3.2 we obtain $u = u_\beta$. Then, we take $u = u_\beta$ in (2.18) and we use the initial condition (2.20) to deduce that β is a solution of Problem $P_{2\beta}$. Finally, using Lemma 3.5, we get $\beta = \beta^*$ and therefore (u_{β^*}, β^*) is a unique solution of Problem P_2 which satisfies (3.1), (3.2).

References

[1] L.-E. Andersson, *Existence result for quasistatic contact problem with Coulomb friction*, Appl. Math. Optimiz. **42** (2000), 169–202.

[2] M. Barboteu, X. Cheng, and M. Sofonea, *Analysis of a contact problem with unilateral constraint and slip-dependent*, Math. Mech. Sol. (2015), to appear.

[3] L. Cangémi, *Frottement et adhérence: modèle, traitement numérique et application à l'interface fibre/matrice*, Ph. D. Thesis, Univ. Méditerranée, Aix Marseille I, 1997.

[4] O. Chau, J. R. Fernandez, M. Shillor, and M. Sofonea, *Variational and numerical analysis of a quasistatic viscoelastic contact problem with adhesion*, J. Comput. Appl. Math. **159** (2003), 431–465.

[5] O. Chau, M. Shillor, and M. Sofonea, *Dynamic frictionless contact with adhesion*, J. Appl. Math. Phys. (ZAMP) **55** (2004), 32–47.

[6] M. Cocu , E. Pratt, and M. Raous, *Formulation and approximation of quasistatic frictional contact*, Int. J. Engng. Sc. **34**, no. 7 (1996), 783–798.

[7] M. Cocou and R. Rocca, *Existence results for unilateral quasistatic contact problems with friction and adhesion*, Math. Model. Num. Anal. **34** (2000), 981–1001.

[8] M. Cocou, M. Schyvre, M. Raous, *A dynamic unilateral contact problem with adhesion and friction in viscoelasticity*, Z. Angew. Math. Phys. **61** (2010), 721–743.

[9] S. Drabla and Z. Zellagui, *Analysis of a electro-elastic contact problem with friction and adhesion*, Studia Univ. "Babes-Bolyai", Mathematica **54**, no. 1 (2009).

[10] G. Duvaut, *Equilibre d'un solide élastique,* C. R. Acad. Sc. Paris Série A **290** (1980), 263–265.

[11] G. Duvaut and J.-L. Lions, *Les inéquations en mécanique et en physique*, Dunod, Paris 1972.

[12] C. Eck, J. Jarušek, and M. Krbec, *Unilateral Contact Problems. Variational Methods and Existence Theorems. Monographs and Textbooks*, in: Pure and Applied Math. **270**, ed. Chapman and Hall / CRC (Taylor and Francis Group), Boca Raton-London-New York-Singapore 2005.

[13] J. R. Fernandez, M. Shillor, and M. Sofonea, *Analysis and numerical simulations of a dynamic contact problem with adhesion*, Math. Comput. Modelling **37** (2003), 1317–1333.

[14] M. Frémond, *Adhérence des solides*, J. Mécanique Théorique et Appliquée **6** (1987), 383–407.

[15] M. Frémond, *Equilibre des structures qui adhèrent à leur support*, C. R. Acad. Sci. Paris **295**, série II (1982), 913–916.

[16] M. Frémond, *Non Smooth Thermomechanics*, Springer, Berlin 2002.

[17] I. R. Ionescu and Q.-L. Nguyen, *Dynamic contact problems with slip dependent friction in viscoelasticity*, Int. J. Appl. Math. Comput. Sci. **12** (2002), 71–80.

[18] J. Jarusěk and M. Sofonea, *On the solvability of dynamic elastic-visco-plastic contact problems*, Zeitschrift fur Angewandte Mathematik and Mechanik (ZAMM), **88**, no. 1 (2008), 3–22.

[19] J. Jarusěk and M. Sofonea, *On the of dynamic elastic-visco-plastic contact problems with adhesion*, Annals of AOSR, Series on Mathematics and its Applications **1** (2009), 191–214.

[20] S. A. Nassar, T. Andrews, S. Kruk, and M. Shillor; *Modelling and simulations of a bonded rod*, Math. Comput. Modelling **42** (2005), 553–572.

[21] G. J. Perrin, J. R. Rice, and G. Zheng, *Self-healing slip pulse on a frictional surface*, J. Mech. Phys. Solids **43** (1995), 1461–1495.

[22] M. Raous, L. Cangémi, and M. Cocu, *A consistent model coupling adhesion, friction, and unilateral contact*, Comput. Meth. Appl. Mech. Engng. **177** (1999), 383–399.

[23] R. Rocca, *Analyse et numérique de problèmes quasi-statiques de contact avec frottement local de Coulomb en élasticité*, Thèse, Aix. Marseille **1** (2005).

[24] J. Rojek and J. J. Telega, *Contact problems with friction, adhesion and wear in orthopeadic biomechanics. I: General developements*, J. Theor. Appl. Mech. **39** (2001), 655–677.

[25] C. H. Scholz, *The Mechanics of Earthquakes and Faulting*, Cambridge University Press, Cambridge 1990.

[26] M. Shillor, M. Sofonea, and J. J. Telega, *Models and variational analysis of quasistatic contact,* Lecture Notes Physics, **655**, Springer, Berlin 2004.

[27] M. Sofonea, W. Han, and M. Shillor, *Analysis and Approximation of Contact Problems with Adhesion or Damage*, Pure and Applied Mathematics **276**, Chapman and Hall / CRC Press, Boca Raton, Florida 2006.

[28] M. Sofonea and T. V. Hoarau-Mantel, *Elastic frictionless contact problems with adhesion*, Adv. Math. Sci. Appl. **15**, no. 1 (2005), 49–68.

[29] M. Sofonea and A. Matei, *Variational inequalities with applications*, Advances in Mathematics and Mechanics **18**, Springer, New York 2009.

[30] M. Sofonea and A. Matei, *An elastic contact problem with adhesion and normal compliance*, Journal of Applied Analysis **12**, no. 1 (2006), 19–36.

[31] A. Touzaline, *Analysis of a frictional contact problem with adhesion for nonlinear elastic materials I*, Bull. Soc. Sci. Lettres Łódź Ser. Rech. Déform. **56** (2008), 61–74.

[32] A. Touzaline, *Frictionless contact problem with finite penetration for elastic materials*, Ann. Pol. Math. **98**, no. 1 (2010), 23–38.

[33] A. Touzaline, *A study of a unilateral and adhesive contact problem with normal compliance,* Appl. Math. (Warsaw) **41** (2014), 385–402.

[34] A. Touzaline, *A quasistatic unilateral contact problem with slip dependent coefficient of friction for nonlinear elastic materials,* Electronic Journal of Differential Equations **2006** (2006), 1–14.

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ANALIZA WARIACYJNA ZAGADNIENIA KONTAKTU POWIERZCHNI Z PRZYLEGANIEM I TARCIEM ZALEŻNYM OD PRZEMIESZCZENIA POŚLIZGOWEGO

S t r e s z c z e n i e

Celem pracy jest zbadanie statycznego kontaktu między elastycznym ciałem a podłożem. Zależności opisujące kontakt są nieliniowe, a sam kontakt modelowany jest za pomocą jednostronnych więzów i odkształcenia normalnego związanych z prawem suchego tarcia Coulomba w wersji z poślizgiem. Przyleganie pomiędzy stykającymi się powierzchniami zostało uwzględnione i wymodelowane za pomocą pola wiążącego, którego zmienność opisana została równaniami różniczkowymi 1. rzędu. Zaproponowano wariacyjne sformułowanie mechanicznego zagadnienia i wykazano istnienie oraz jednoznaczność rozwiązania. Technika dowodzenia oparta została na zależnych od czasu nierównościach wariacyjnych, równaniach różniczkowuch i twierdzeniu Banacha o punkcie stałym.

Słowa kluczowe: elastyczność, odkształcenie normalne, adhezja, tarcie, więzy jednostronne

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Arezki Touzaline

A FRICTIONAL CONTACT PROBLEM WITH UNILATERAL CONSTRAINT AND NORMAL COMPLIANCE

Summary

We consider a mathematical model which describes the equilibrium of a nonlinear elastic body in frictional contact with a foundation. The contact is modelled with normal compliance and unilateral constraints, associated with Coulomb's law of dry friction. We establish a variational formulation of the mechanical problem and prove an existence and uniqueness result if the coefficient of friction is small enough. The technique of the proof is based on arguments of elliptic quasivariational inequalities and Banach fixed-point theorem. We also study a penalized and regularized problem which admits at least one solution and prove its convergence to the solution of the model when the penalization and regularization parameter tends to zero.

Keywords and phrases: elastic, normal compliance, friction, unilateral constraint

1. Introduction

Contact problems involving deformable bodies are quite frequent in industry as well as in daily life and play an important role in structural and mechanical systems. Contact processes involve complicated surface phenomena, and are modelled with highly nonlinear initial boundary value problems. Taking into account various contact conditions associated with more and more complex behavior laws leads to the introduction of new and non standard models, expressed by the aid of evolution variational inequalities. An early attempt to study contact problems within the framework of variational inequalities was made in [7]. The mathematical, mechanical and numerical state of the art can be found in [10] where we find detailed mathematical and numerical studies of the contact problems. We recall that unilateral contact problems involving Signorini's condition have been studied by several authors, see for instance the papers [1, 3–6, 8–11, 13, 14] and the references therein.

In this paper, we study a mathematical model which describes a static contact with normal compliance, unilateral constraint and slip-dependent friction law for elastic materials. We present a weak formulation of the problem, then we state and prove an existence and uniqueness of the solution if the coefficient of friction is small enough. We also consider a penalized and regularized problem which has at least one solution and prove its convergence to the solution of the model when the parameter of penalization and regularization tends to zero.

The paper is structured as follows. In Section 2 we present some notations and give the variational formulation. In Section 3 we state and prove our main existence and uniqueness result, Theorem 3.1. In section 4 we establish a convergence result, Theorem 4.2. Also, we study the mechanical interpretation on the contact surface, Theorem 4.6.

2. Problem statement and variational formulation

We consider a nonlinear elastic body which occupies a domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ and assume that its boundary Γ is regular and partitioned into three measurable and disjoint parts $\Gamma_1, \Gamma_2, \Gamma_3$ such that $meas\,(\Gamma_1) > 0$. The body is acted upon by a volume force of density f_1 in Ω and a surface traction of density f_2 on Γ_2. On Γ_3 the body is in unilateral contact following the Coulomb's friction law with a foundation.

Then, the classical formulation of the mechanical problem in terms of displacement field is written as follows.

Problem P_1. Find a displacement field $u : \Omega \to \mathbb{R}^d$ such that

$$div\sigma\,(u) = -f_1 \text{ in } \Omega, \tag{2.1}$$

$$\sigma\,(u) = F\varepsilon\,(u) \quad \text{in } \Omega, \tag{2.2}$$

$$u = 0 \quad \text{on } \Gamma_1, \tag{2.3}$$

$$\sigma\nu = f_2 \quad \text{on } \Gamma_2, \tag{2.4}$$

$$\left.\begin{aligned} & u_\nu \leq g,\ \sigma_\nu + p\,(u_\nu) \leq 0 \\ & (\sigma_\nu + p\,(u_\nu))\,(u_\nu - g) = 0 \end{aligned}\right\} \text{ on } \Gamma_3, \tag{2.5}$$

$$\left.\begin{aligned} & |\sigma_\tau| \leq \mu p\,(u_\nu) \\ & |\sigma_\tau| < \mu p\,(u_\nu) \Longrightarrow u_\tau = 0 \\ & |\sigma_\tau| = \mu p\,(u_\nu) \Longrightarrow \\ & \exists \lambda \geq 0;\ \sigma_\tau = -\lambda u_\tau \end{aligned}\right\} \text{on } \Gamma_3. \tag{2.6}$$

Equation (2.1) represents the equilibrium equation. Equation (2.2) is the elastic constitutive law of the material in which $\sigma = \sigma(u)$ denotes the stress field, F is a given function and $\varepsilon(u)$ denotes the strain tensor. Relations (2.3) and (2.4) are the displacement and traction boundary conditions, respectively, in which ν denotes the unit outward normal vector on Γ and $\sigma\nu$ represents the Cauchy stress vector. The condition (2.5) represents the unilateral contact with normal compliance p which satisfies the assumption below (2.14). We denote by the positive constant g the maximum value of the penetration. When $u_\nu < 0$, there is separation between the body and the foundation, then the condition (2.5) combined with (2.14) shows that $\sigma_\nu = -p_\nu(u_\nu)$ and it does not exeed the value $L_p g$. When $g > 0$, the body may interpenetrate into the foundation, but the penetration is limited that is $u_\nu \leq g$. In this case of penetration (i.e. $u_\nu \geq 0$), when $0 \leq u_\nu < g$ then $-\sigma_\nu = p(u_\nu)$ which means that the reaction of the foundation is uniquely determined by the normal displacement and $\sigma_\nu \leq 0$. Since p is an increasing function then the reaction of the foundation is increasing with the penetration and when $u_\nu = g$, then $-\sigma_\nu \geq p(g)$ and σ_ν is not uniquely determined. When $g > 0$ and $p = 0$, the condition (2.5) becomes the Signorini contact condition with a gap,

$$u_\nu \leq g, \ \sigma_\nu \leq 0, \ \sigma_\nu(u_\nu - g) = 0.$$

When $g = 0$, the condition (2.5) combined with assumption (2.14) becomes the Signorini contact condition with a zero gap, given by

$$u_\nu \leq 0, \ \sigma_\nu \leq 0, \ \sigma_\nu u_\nu = 0.$$

We denote by S_d the space of second order symmetric tensors on $\mathbb{R}^d$ $(d = 2, 3)$ and $|.|$ represents the Euclidean norm on $\mathbb{R}^d$ and S_d. Thus, for every $u, v \in \mathbb{R}^d$,

$$u.v = u_i v_i, \quad |v| = (v.v)^{\frac{1}{2}},$$

and for every $\sigma, \tau \in S_d$,

$$\sigma.\tau = \sigma_{ij}\tau_{ij}, \quad |\tau| = (\tau.\tau)^{\frac{1}{2}}.$$

Here and below, the indices i and j run between 1 and d and the summation convention over repeated indices is adopted. Now, to proceed with the variational formulation, we need the following function spaces:

$$H = \left(L^2(\Omega)\right)^d, \ H_1 = \left(H^1(\Omega)\right)^d, \ Q = \left\{\tau = (\tau_{ij}); \ \tau_{ij} = \tau_{ji} \in L^2(\Omega)\right\},$$
$$Q_1 = \{\tau \in Q; \ div\tau \in H\}.$$

Note that H and Q are real Hilbert spaces endowed with the respective canonical inner products

$$(u, v)_H = \int_\Omega u_i v_i dx, \quad (\sigma, \tau)_Q = \int_\Omega \sigma_{ij}\tau_{ij} dx.$$

The strain tensor is

$$\varepsilon(u) = (\varepsilon_{ij}(u)), \text{ where } \varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i});$$

$div\sigma = (\sigma_{ij,j})$ is the divergence of σ. For every $v \in H_1$ we denote by v_ν and v_τ the normal and tangential components of v on the boundary Γ given by

$$v_\nu = v.\nu, \quad v_\tau = v - v_\nu \nu.$$

We also denote by σ_ν and σ_τ the normal and the tangential traces of a function $\sigma \in Q_1$, and when σ is a regular function then

$$\sigma_\nu = (\sigma\nu).\nu, \quad \sigma_\tau = \sigma\nu - \sigma_\nu \nu,$$

and the following Green's formula holds:

$$(\sigma, \varepsilon(v))_Q + (div\sigma, v)_H = \int_\Gamma \sigma\nu.vda \quad \forall v \in H_1,$$

where da is the surface measure element. Now, let V be the closed subspace of H_1 defined by

$$V = \{v \in H_1 : v = 0 \text{ on } \Gamma_1\},$$

and let the convex subset of admissible displacements given by

$$K = \{v \in V : v_\nu \leq g \ a.e. \text{ on } \Gamma_3\}.$$

Since $meas(\Gamma_1) > 0$, the following Korn's inequality holds [7],

$$\|\varepsilon(v)\|_Q \geq c_\Omega \|v\|_{H_1} \quad \forall v \in V, \tag{2.9}$$

where $c_\Omega > 0$ is a constant which depends only on Ω and Γ_1. We equip V with the inner product

$$(u, v)_V = (\varepsilon(u), \varepsilon(v))_Q$$

and $\|.\|_V$ is the associated norm. It follows from Korn's inequality (2.9) that the norms $\|.\|_{H_1}$ and $\|.\|_V$ are equivalent on V. Then $(V, \|.\|_V)$ is a real Hilbert space. Moreover by Sobolev's trace theorem, there exists $d_\Omega > 0$ which only depends on the domain Ω, Γ_1 and Γ_3 such that

$$\|v\|_{(L^2(\Gamma_3))^d} \leq d_\Omega \|v\|_V \quad \forall v \in V. \tag{2.10}$$

We suppose that the body forces and surface tractions have the regularity

$$f_1 \in H, \quad f_2 \in (L^2(\Gamma_2))^d \tag{2.11}$$

and let f the element of V defined by

$$(f, v)_V = \int_\Omega f_1.vdx + \int_{\Gamma_2} f_2.vda \quad \forall v \in V.$$

In the study of the mechanical problem P_1 we assume that the nonlinear elasticity operator $F : \Omega \times S_d \to S_d$ satisfies:

$$(2.12)\qquad \left.\begin{array}{l} (a) \text{ There exists } M>0 \text{ such that} \\ \quad |F(x,\varepsilon_1)-F(x,\varepsilon_2)| \le M\,|\varepsilon_1-\varepsilon_2| \quad \forall\, \varepsilon_1,\varepsilon_2 \in S_d, \\ \quad a.e.\ x\in\Omega; \\ \\ (b) \text{ there exists } m>0 \text{ such that} \\ \quad (F(x,\varepsilon_1)-F(x,\varepsilon_2)).(\varepsilon_1-\varepsilon_2) \ge m\,|\varepsilon_1-\varepsilon_2|^2, \\ \quad \forall \varepsilon_1,\varepsilon_2\in S_d,\ a.e.\ x\in\Omega; \\ \\ (c) \text{ the mapping } x\to F(x,\varepsilon) \text{ is Lebesgue measurable on } \Omega, \\ \quad \text{for any } \varepsilon\in S_d; \\ \\ (d)\ F(x,0)=0 \text{ for } a.e.\ x\in\Omega. \end{array}\right\}$$

Next we define the functional $j_c : V\times V\to\mathbb{R}$ by

$$j_c(u,v)=\int_{\Gamma_3} p(u_\nu)v_\nu da,\ \forall (u,v)\in V\times V$$

and the functional $j_f : V\times V\to\mathbb{R}_+$ by

$$j_f(u,v)=\int_{\Gamma_3}\mu p(u_\nu)\,|v_\tau|\,da \qquad \forall (u,v)\in V\times V,$$

where the coefficient of friction μ satisfies

$$(2.13)\qquad \mu\in L^\infty(\Gamma_3) \text{ and } \mu\ge 0\ a.e. \text{ on } \Gamma_3.$$

We assume that the normal compliance function $p:\Gamma_3\times\mathbb{R}\to\mathbb{R}_+$ satisfies:

$$(2.14)\qquad \left\{\begin{array}{l} (a)\ \text{There exists } L_p>0 \text{ such that} \\ \quad |p(x,r_1)-p(x,r_2)|\le L_p\,|r_1-r_2| \\ \quad \forall r_1,\, r_2\in\mathbb{R},\ a.e.\ x\in\Gamma_3; \\ \\ (b)\ (p(x,r_1)-p(x,r_2))(r_1-r_2)\ge 0 \\ \quad \forall r_1,\, r_2\in\mathbb{R},\ a.e.\ x\in\Gamma_3; \\ \\ (c)\ \text{the mapping } x\to p(x,r) \text{ is measurable on } \Gamma_3, \text{ for any } r\in\mathbb{R}; \\ \\ (d)\ p(x,r)=0\ \forall r\le 0,\ a.e.\ x\in\Gamma_3. \end{array}\right.$$

Now let us define

$$H^{\frac12}(\Gamma_3)=\left\{\mu\,|_{\Gamma_3} : \mu\in H^{\frac12}(\Gamma),\ \mu=0 \text{ on } \Gamma_1\right\}$$

equipped with the norm of $H^{\frac12}(\Gamma)$. $\langle .,.\rangle$ shall denote the duality pairing on $H^{\frac12}(\Gamma_3)$, $H^{-\frac12}(\Gamma_3)$ and $[.,.]$ shall denote the duality pairing on $(H^{\frac12}(\Gamma_3))^d$, $(H^{-\frac12}(\Gamma_3))^d$.

Finally, by a standard procedure based on the Green formula, we obtain the following variational formulation of Problem P_1.

Problem P_2. Find a displacement field $u \in K$ such that

$$(2.15)\qquad \begin{aligned} &(F\varepsilon(u), \varepsilon(v-u))_Q + j_c(u, v-u) \\ &+ j_f(u,v) - j_f(u,u) \geq (f, v-u)_V \quad \forall\, v \in K. \end{aligned}$$

3. Existence of solution for problem P_2

In the study of the problem P_2 we have the following existence and uniqueness result.

Theorem 3.1. *Let* (2.11), (2.12), (2.13) *and* (2.14) *hold. Then Problem* P_2 *has a unique solution if*

$$\|\mu\|_{L^\infty(\Gamma_3)} < m/L_p d_\Omega^2.$$

The proof of Theorem 3.1 will be established in several steps. Indeed, in the first step for a given $\eta \in K$, we consider the following intermediate problem.

Problem P_η. Find $u_\eta \in K$ such that

$$(3.1)\qquad \begin{aligned} &(F\varepsilon(u_\eta), \varepsilon(v-u_\eta))_Q + j_c(u_\eta, v-u_\eta) + j_f(\eta, v) \\ &- j_f(\eta, u_\eta) \geq (f, v-u_\eta)_V \quad \forall v \in K. \end{aligned}$$

Lemma 3.3. *Problem* P_η *has a unique solution.*

Proof. Let the operator $A : V \to V$ defined by

$$(Au, v)_V = (F\varepsilon(u), \varepsilon(v))_Q + j_c(u, v), \ \forall u, v \in V.$$

We use (2.10), (2.12a), (2.12b), (2.14a) and (2.14b) to show that the operator A is strongly monotone and Lipschitz continuous; the functional $j(\eta, .) : K \to \mathbb{R}_+$ is a continuous seminorm; then by a standard existence and uniqueness result for elliptic variational inequalities (see [2]), it follows that there exists a unique element $u_\eta \in K$ which satisfies the inequality (3.1) since K is a non-empty, closed convex subset of V. $\square$

Now, in the second step, we consider the map $T : K \to K$ defined as

$$T(\eta) = u_\eta.$$

We have the following lemma.

Lemma 3.4. *The map* T *has a unique fixed point* η^* *and* u_{η^*} *is a unique solution of Problem* P_2.

Proof. Let $\eta_1, \eta_2 \in K$. In inequality (3.1) satisfied by u_{η_1} take $v = u_{\eta_2}$ and also in the same inequality satisfied by u_{η_2} take $v = u_{\eta_1}$. Then, using (2.10), (2.12b) and (2.14b), it follows after adding the resulting inequalities that

$$\|T(\eta_1) - T(\eta_2)\|_V \leq \frac{\|\mu\|_{L^\infty(\Gamma_3)} L_p d_\Omega^2}{m} \|\eta_1 - \eta_2\|_V \,, \; \forall \eta_1, \eta_2 \in K.$$

Then, for

$$\|\mu\|_{L^\infty(\Gamma_3)} L_p d_\Omega^2)/m < 1,$$

the map T is a contraction; so it has a unique fixed point η^* and u_{η^*} is a unique solution of Problem P_2. Next, denote $u_{\eta^*} = u$. □

4. A convergence result

In this section we consider a frictional contact problem with normal compliance where the penetration is unlimited. The contact conditions (2.5) and (2.6) are replaced respectively on Γ_3 by

$$-\sigma_{\delta\nu} = \frac{1}{\delta}(u_{\delta\nu} - g)_+ + p(u_{\delta\nu})$$

and

$$\sigma_{\delta\tau} = -\mu p(u_{\delta\nu}) \frac{u_{\delta\tau}}{\sqrt{|u_{\delta\tau}|^2 + \delta^2}} \,,$$

where $\delta > 0$ is a penalization and regularization parameter. Then we define the penalized and regularized problem as follows.

Problem P_δ. Find a displacement field $u_\delta : \Omega \to \mathbb{R}^d$ such that

$$div\sigma(u_\delta) = -f_1 \text{ in } \Omega,$$

$$\sigma(u_\delta) = F\varepsilon(u_\delta) \quad \text{in } \Omega,$$

$$u_\delta = 0 \quad \text{ on } \Gamma_1,$$

$$\sigma_\delta \nu = f_2 \quad \text{ on } \Gamma_2,$$

$$-\sigma_{\delta\nu} = \frac{1}{\delta}(u_{\delta\nu} - g)_+ + p(u_{\delta\nu}) \text{ on } \Gamma_3,$$

$$\sigma_{\delta\tau} = -\mu p(u_{\delta\nu}) \frac{u_{\delta\tau}}{\sqrt{|u_{\delta\tau}|^2 + \delta^2}} \text{ on } \Gamma_3.$$

We denote $\sigma_\delta = \sigma(u_\delta)$ and we recall that $1/\delta$ is interpreted as the stiffness coefficient of the foundation. We understand that when δ is small, the reaction of the foundation to the penetration is important; also when δ is large then the reaction of the foundation to the penetration is weaker. We study the behavior of the solution as $\delta \to 0$ and prove that in the limit we obtain the solution of frictional contact problem with normal compliance and finite penetration. In the next we define the functionals $j_{c\delta} : V \times V \to \mathbb{R}$ and $j_{f\delta} : V \times V \to \mathbb{R}$

by

$$j_{c\delta}(u,v)=\int_{\Gamma_3}\left(\frac{1}{\delta}(u_\nu-g)_+ +p(u_\nu)\right)v_\nu da,\ \forall (u,v)\in V\times V,$$
$$j_{f\delta}(u,v)=\int_{\Gamma_3}\mu p(u_\nu)\frac{u_\tau}{\sqrt{|u_\tau|^2+\delta^2}}v_\tau da,\ \forall (u,v)\in V\times V.$$

With these notations, the variational formulation of the penalized and regularized problem with frictional contact is the following.

Problem P_δ. Find a displacement field $u_\delta\in V$ such that

$$(F\varepsilon(u_\delta),\varepsilon(v))_Q+j_{c\delta}(u_\delta,v)+j_{f\delta}(u_\delta,v)=(f,v)_V\,,\ \forall v\in V. \tag{4.1}$$

We have the following result.

Theorem 4.1. *Problem P_δ has at least one solution.*

Proof. As in [8], to prove Theorem 4.1 we use the arguments of pseudomonotone operators. In fact, we define the operators B, C, $D:V\to V'$ by

$$\langle Bu,v\rangle_{V'\times V}=(F\varepsilon(u),\varepsilon(v))_Q,\ \forall u,v\in V,$$
$$\langle Cu,v\rangle_{V'\times V}=j_{c\delta}(u,v)\,,\ \forall u,v\in V,$$
$$\langle Du,v\rangle_{V'\times V}=j_{f\delta}(u,v)\,,\ \forall u,v\in V.$$

We use the assumption (2.12) to see that the operator B is bounded and elleptic. Indeed, for all $u,v\in V$, the following holds:

$$\langle Bu,v\rangle_{V'\times V}\le M\|u\|_V\|v\|_V \text{ and } \langle Bv,v\rangle_{V'\times V}\ge m\|v\|_V^2\,.$$

We also use (2.13), the compact embedding $H^{\frac{1}{2}}(\Gamma_3)\hookrightarrow L^2(\Gamma_3)$ and the Lebesgue dominated convergence, to show that the operators C and D are completely continuous.and bounded for each δ. Moreover, we have

$$\langle Cv,v\rangle_{V'\times V}\ge 0 \text{ and } \langle Dv,v\rangle_{V'\times V}\ge 0\ \ \forall v\in V.$$

Then the operator $E=B+C+D$ is pseudomonotone, bounded and coercive. Consequently, we deduce that the equation (4.1) has at least one solution $u_\delta\in V$.

Now, we study the convergence of the solution u_δ, as $\delta\to 0$ in the following theorem.

Theorem 4.2. *Assume that* (2.12), (2.13) *and* (2.14) *hold. Then we have the following strong convergence:*

$$\lim_{\delta\to 0}\|u_\delta-u\|_V=0. \tag{4.2}$$

The proof is carried out in several steps. In the first step, we show the following lemma.

Lemma 4.3. *There exists $\bar{u}\in K$ such that after passing to a subsequence still denoted (u_δ), we have*

$$u_\delta\to\bar{u} \text{ weakly in } V. \tag{4.3}$$

Proof. Take $v = u_\delta$ in (4.1), we have

$$(F\varepsilon(u_\delta), \varepsilon(u_\delta))_Q + j_{c\delta}(u_\delta, u_\delta) + j_{f\delta}(u_\delta, u_\delta) = (f, u_\delta)_V . \tag{4.4}$$

As $j_{c\delta}(u_\delta, u_\delta) \geq 0$ and $j_{f\delta}(u_\delta, u_\delta) \geq 0$, it is easy to deduce by (2.12) (b) that

$$\|u_\delta\|_V \leq \|f\|_V / m.$$

Then there exists an element $\bar{u} \in V$ and a subsequence still denoted u_δ such that

$$u_\delta \to \bar{u} \text{ weakly on } V.$$

On the other hand from (4.4), we get the inequality

$$\int_{\Gamma_3} \left(\frac{u_{\delta\nu} - g}{\delta}\right)_+ (u_{\delta\nu} - g)\, da \leq (f, u_\delta)_V ,$$

which implies that

$$\left\|(u_{\delta\nu} - g)_+\right\|^2_{L^2(\Gamma_3)} \leq \delta \, \|f\|^2_V \ /m.$$

Then it follows that

$$\left\|(\bar{u}_\nu - g)_+\right\|_{L^2(\Gamma_3)} \leq \liminf_{\delta\to 0} \left\|(u_{\delta\nu} - g)_+\right\|_{L^2(\Gamma_3)} = 0. \tag{4.5}$$

Therefore we conclude by (4.5) that $(\bar{u}_\nu - g)_+ = 0$, i-e. $\bar{u}_\nu \leq g$ *a.e.* on Γ_3 and then $\bar{u} \in K$. □

Next, we prove the following lemma.

Lemma 4.4. *We have* $\bar{u} = u$.

Proof. Let $v \in K$ and choose $v - u_\delta$ in (4.1) yields

$$\begin{aligned} &(F\varepsilon(u_\delta), \varepsilon(v - u_\delta))_Q + j_{c\delta}(u_\delta, v - u_\delta) \\ &+ j_{f\delta}(u_\delta, v - u_\delta) \geq (f, v - u_\delta)_V \quad \forall v \in K. \end{aligned} \tag{4.6}$$

Since we have

$$\begin{aligned} &j_{c\delta}(u_\delta, v - u_\delta) \\ &= \textstyle\int_{\Gamma_3} \left(\left(\dfrac{u_{\delta\nu} - g}{\delta}\right)_+ + p(u_{\delta\nu})\right)(v_\nu - u_{\delta\nu})\, da \\ &\leq \textstyle\int_{\Gamma_3} p(u_{\delta\nu})(v_\nu - u_{\delta\nu})\, da, \end{aligned}$$

then we use (2.14a) and the compact imbedding $H^{\frac{1}{2}}(\Gamma_3) \hookrightarrow L^2(\Gamma_3)$ to see that

$$\lim_{\delta\to 0} \textstyle\int_{\Gamma_3} p(u_{\delta\nu})(v_\nu - u_{\delta\nu})\, da = j_c(\bar{u}, v - \bar{u})$$

and

$$\limsup_{\delta\to 0} j_{f\delta}(u_\delta, v - u_\delta) \leq j_f(\bar{u}, v) - j_f(\bar{u}, \bar{u}).$$

On the other hand as in [12], we have

$$\lim_{\delta\to 0}\left(F\varepsilon\left(u_\delta\right),\varepsilon\left(v-u_\delta\right)\right)_Q=\left(F\varepsilon(\bar{u}),\varepsilon\left(v-\bar{u}\right)\right)_Q,$$

therefore, passing to the limit in (4.6) as $\delta\to 0$, we obtain

$$\begin{aligned}&\left(F\varepsilon(\bar{u}),\varepsilon\left(v-\bar{u}\right)\right)_Q+j_c\left(\bar{u},v-\bar{u}\right)+j_f(\bar{u},v)\\ &-j_f(\bar{u},\bar{u})\geq\left(f,v-\bar{u}\right)_V\quad\forall v\in K.\end{aligned}\tag{4.7}$$

Now, we choose $v=u$ in (4.7) and $v=\bar{u}$ in (2.15) and adding the resulting inequalities, we obtain by using the assumption (2.12b) that

$$\begin{aligned}&m\left\|\bar{u}-u\right\|_V^2\leq j_c\left(\bar{u},u-\bar{u}\right)+j_c\left(u,\bar{u}-u\right)\\ &+j_f\left(\bar{u},u\right)-j_f\left(\bar{u},\bar{u}\right)+j_f\left(u,\bar{u}\right)-j_f\left(u,u\right).\end{aligned}$$

Moreover using (2.14b), we see that

$$j_c\left(\bar{u},u-\bar{u}\right)+j_c\left(u,\bar{u}-u\right)\leq 0,$$

and then we deduce that

$$\begin{aligned}&m\left\|\bar{u}-u\right\|_V^2\leq\\ &j_f\left(\bar{u},u\right)-j_f\left(\bar{u},\bar{u}\right)+j_f\left(u,\bar{u}\right)-j_f\left(u,u\right).\end{aligned}$$

On the other hand using (2.10) and (2.14b), we have

$$\begin{aligned}&j_f\left(\bar{u},u\right)-j_f\left(\bar{u},\bar{u}\right)+j_f\left(u,\bar{u}\right)-j_f\left(u,u\right)\leq\\ &L_p d_\Omega^2\left\|\bar{u}-u\right\|_V^2.\end{aligned}$$

Hence we get

$$\left(m-L_p d_\Omega^2\left\|\mu\right\|_{L^\infty(\Gamma_3)}\right)\left\|\bar{u}-u\right\|_V^2\leq 0,$$

and then as $m-L_p d_\Omega^2\left\|\mu\right\|_{L^\infty(\Gamma_3)}>0$, we obtain

$$\bar{u}=u.\tag{4.8}$$

Now, to prove (4.2), we use (2.12b) to see that

$$\begin{aligned}&m\left\|u_\delta-u\right\|_V^2\leq\\ &(F\varepsilon(u_\delta),\varepsilon(u_\delta-u))_Q-(F\varepsilon(u),\varepsilon(u_\delta-u))_Q.\end{aligned}$$

Passing to the limit as $\delta\to 0$ in the above inequality and taking into account that

$$\lim_{\delta\to 0}\Big((F\varepsilon(u_\delta),\varepsilon(u_\delta-u))_Q-(F\varepsilon(u),\varepsilon(u_\delta-u))_Q\Big)=0,$$

we deduce the strong convergence (4.2).

Remark 4.5. We have $\sigma_\delta\to\sigma$ strongly in Q_1. Indeed, we have

$$\left\|\sigma_\delta-\sigma\right\|_{Q_1}=\left\|\sigma_\delta-\sigma\right\|_Q\leq M\left\|u_\delta-u\right\|_V.$$

Therefore, the strong convergence is a consequence of (4.2).

Now, we are interesting to study the mechanical interpretation on the contact surface Γ_3. Then we have the theorem below.

Theorem 4.6. *We have the following convergences:*

$$\text{(4.9)} \qquad i)\ \|\sigma_{\delta\nu}-\sigma_\nu\|_{H^{-\frac{1}{2}}(\Gamma_3)} \to 0,\ \ ii)\ \|\sigma_{\delta\tau}-\sigma_\tau\|_{(H^{-\frac{1}{2}}(\Gamma_3))^d} \to 0,\ as\ \delta\to 0.$$

Proof. The solution u of Problem P_1 satisfies the equality:

$$\text{(4.10)} \qquad (F\varepsilon(u),\varepsilon(v))_Q - \langle\sigma_\nu, v_\nu\rangle - [\sigma_\tau, v_\tau] = (f,v)_V \quad \forall v\in V.$$

Using the equalities (4.1) and (4.10) we deduce the following equality:

$$\text{(4.11)} \qquad (F\varepsilon(u_\delta) - F\varepsilon(u),\varepsilon(v))_Q - \langle\sigma_{\delta\nu}-\sigma_\nu, v_\nu\rangle - [\sigma_{\delta\tau}-\sigma_\tau, v_\tau] = 0\ \forall v\in V.$$

Now, let $v\in V$ such that $v_\tau = 0$ and passing to the limit as $\delta\to 0$ in (4.11), then by (4.2) we get (4.9 i). In the same manner take $v\in V$ such that $v_\nu = 0$, with the same reasoning we get (4.9 ii). □

References

[1] L.-E. Andersson, *Existence result for quasistatic contact problem with Coulomb friction*, Appl. Math. Optimiz. **42** (2000), 169–202.

[2] H. Brezis, *Equations et inéquations non linéaires dans les espaces vectoriels en dualité,* Annales Inst. Fourier **18** (1968), 115–175.

[3] A. Capatina Radoslovescu and M. Cocu, *Internal approximation of quasi-variational*, Numer. Math. **59** (1991), 385–398.

[4] M. Cocu, *Existence of solutions of Signorini problems wih friction,* Int. J. Engng. Sci. **22** (1984), 567–575.

[5] M. Cocu , E. Pratt, and M. Raous, *Formulation and approximation of quasistatic frictional contact*, Int. J. Engng. Sc. **34**, no. 7 (1996), 783–798.

[6] G. Duvaut, *Equilibre d'un solide élastique avec contact unilatéral et frottement de coulomb,* C. R. Acad. Sc. Paris Série A, **290** (1980), 263–265.

[7] G. Duvaut and J.-L.Lions, *Les Inéquations en Mécanique et en Physique*, Dunod, Paris 1972.

[8] C. Eck, J. Jarušek, and M. Krbec, *Unilateral Contact Problems. Variational Methods and Existence Theorems,* Monographs and Textbooks in Pure and Applied Math. no. 270, ed. Chapman and Hall / CRC (Taylor and Francis Group), Boca Raton-London-New York-Singapore 2005.

[9] J. Jarusěk and M. Sofonea, *On the solvability of dynamic elastic-visco-plastic contact problems*, Zeitschrift fur Angewandte Mathematik and Mechanik (ZAMM) **88**, no. 1 (2008),3–22.

[10] N. Kikuchi and T. J. Oden, *Contact Problems in Elasticity: A Study of Variational Inequalities and Finite Element Methods*, SIAM, Philadelphia 1988

[11] R. Rocca, *Analyse et numérique de problèmes quasi-statiques de contact avec frottement local de Coulomb en élasticit é*, Thèse, Aix. Marseille 1, 2005.

[12] M. Rochdi and B. Teniou, *Frictional contact problems for nonlinear elastic materials,* Appl. Math. Comp. Sci. **8,** no.3 (1998), 491–504.

[13] A. Touzaline and D. Teniou, *A quasistatic unilateral and frictional contact problem for nonlinear elastic materials*, Mathematical Modelling and Analysis **12**, no. 4 (2007), 497–514.

[14] A. Touzaline, *Study of a contact problem with normal compliance and nonlocal friction*, Appl. Math. (Warsaw) **39** (2012), 43–55.

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PROBLEM TARCIA W ASPEKCIE JEDNOSTRONNYCH WIĘZÓW I ODKSZTAŁCENIA NORMALNEGO

S t r e s z c z e n i e

W pracy został przedstawiony matematyczny model opisujący stan równowagi nieliniowego ciała elastycznego związanego z podłożem poprzez tarcie. Kontakt wymodelowano z uwzględnieniem jednostronnych więzów i odkształcenia normalnego, związanych z prawem Coulomba o suchym tarciu. Autorzy przedstawiają wariacyjne sformułowanie problemu mechanicznego i dowodzą jednoznaczności wyników przy założeniu, że współczynnik tarcia jest odpowiednio mały. Dowód oparty jest na argumentach wynikających z rozważania eliptycznych nierówności quasi-wariacyjnych oraz twierdzenia Banacha o punkcie stałym.

Słowa kluczowe: elastyczność, odkształcenie normalne, tarcie, więzy jednostronne

Rapporteurs – Referees

CONTENU DU VOLUME LXV, no. 2